Total Reflection of Relic Neutrinos from Material Targets

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Abstract

Neutrinos from the Big Bang are expected to be the most abundant particles in the universe together with photons of the Cosmic Microwave Background. However, at present all the evidences for existence of the relic neutrinos rely on the cosmological observations such as light element abundances, Cosmic Microwave Background anisotropies and large scale structures of the universe and they are somewhat indirect to prove the existence of the relic neutrinos. Although the possibility of more directly detecting relic neutrinos through laboratory experiments has been discussed, all expected laboratory effects of relic neutrinos upon materials are far from observability awaiting future technological advances to reach the necessary sensitivity or some genius, so far unexplored, methods to detect relic neutrinos.

In this paper we study whether relic neutrinos can be totally reflected from material targets and some related problems such as flavor mixing, helicity flip of neutrinos due to total reflection and negative energy contribution to suppress the total reflection in case of very high repulsive potential. Although there is no immediate use of the total reflection of relic neutrinos for their observation at present, we hope that the total reflection may turn out to become useful for making density gradients of relic neutrinos in space and asymmetry between neutrino and antineutrino densities so that with future technological advances the detection of relic neutrinos through laboratory experiments will become feasible in remote future.

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1. Introduction

According to the standard Big Bang model of the universe, neutrinos were decoupled from other particles when the universe was about one second old and its plasma temperature was of the order of one MeV; they have remained as relic neutrinos up to the present.

There are cosmological observations which are confirming or consistent with the existence of relic neutrinos: observations of light element abundances which are related with the Big Bang Nucleosynthesis, and Cosmic Microwave Background anisotropies together with large scale structures of the universe which are sensitive to the masses of relic neutrinos. More direct observation of these relic neutrinos has been desired and much effort has been paid to consider observational methods. They include a proposal to observe spin precession of electrons [1] assuming the asymmetric abundance of neutrinos and antineutrinos, proposals[2] to observe the target effects of relic neutrinos in the production of the neutral weak boson $\nu+\bar{\nu}\to \mathrm{Z}^0$ by very high energy cosmic ray neutrinos assuming the existence of such cosmic ray neutrinos with $E>10^{22}eV$, and proposals[3] to observe the modification of the end point of beta decay $A\to B+e^-+\bar{\nu}_e$ to be made by the suppression of the antineutrino emission due to Pauli exclusion principle and/or by the absorption of the relic neutrino $\nu_e+A\to B+e^-$, using beta decay source nuclei at rest or accelerated. Among the various proposals, the mechanical method[4] to see the recoil of relic neutrino scattering are very attractive, since it takes advantage of enhancement due to coherent summation of amplitudes of the neutrino scattering off many atoms in the target, for example, there are proposals[4] to observe the 'wind force' of relic neutrinos produced by the Sun's peculiar motion through the Galaxy and modified by the Earth's motion, using inhomogeneous target with a special inhomogeneity size comparable to or a little less than neutrino wave lengths, though it is still a great challenge to detect such small effects demanding new ideas and technologies.

In the following we study the total reflection of relic neutrinos on the material target surface[4] and examine related problems such as the effects of flavor mixing, helicity flip and transition to the negative energy states of neutrinos in the total reflection process. Though we have at present no immediate application of the total reflection for observation of relic neutrinos, we hope that it may turn out to become useful in the detection of relic neutrinos if we could have further technological advances in a remote future.

Let us first briefly review the properties of relic neutrinos. Densities of the relic neutrinos are usually expected to be uniform in space, and equal for the neutrinos and antineutrinos. They are about $\sim 56 cm^{-3}$ for each neutrino or antineutrino of the three kinds of the neutrino flavors:

$$\rho_{\nu_\alpha} = \rho_{\bar{\nu}_\alpha} \approx 56 cm^{-3} \qquad (\nu_\alpha = \nu_e, \nu_\mu, \text{ or } \nu_\tau) \tag{1}$$

though a possibility of $\rho_{\nu_\alpha} \neq \rho_{\bar{\nu}_\alpha}$ will be briefly mentioned below. The momentum distribution of the relic neutrinos at the decoupling temperature T_D ($\sim 1 MeV$) is given by the thermal Fermi-Dirac distribution function $F_D(p)$ for the neutrino momentum p,

$$F_D(p) = 1/[\exp\{(\sqrt{p^2+m^2} - \mu_D)/T_D\}+1] \ , \tag{2}$$

where m is the neutrino mass and μ_D is the chemical potential of the neutrino.

After the decoupling of neutrinos from other particles, the momentum undergoes redshift to the present one $p \to p/\eta$ with the scaling parameter $\eta = 1+z$ due to the expansion of the universe. The present momentum spectrum has become

$$F(p) = F_D(\eta p) = 1/\left(\exp[(\sqrt{p^2+m_{eff}{}^2} - \mu)/T]+1\right) \tag{3}$$

with $T = T_D/\eta,\ \mu = \mu_D/\eta$ and $m_{eff} = m/\eta$.

The present neutrino spectrum is thus expressed by the present temperature T, the present chemical potential μ, and the effective neutrino mass m_{eff}. The temperature T is theoretically related[5] to the present Cosmic Microwave Background temperature $T_\gamma \approx 2.7325°K$ through $T = (4/11)^{1/3} T_\gamma$, which is due to the annihilation $e^+e^- \to 2\gamma$ after the neutrino decoupling and gives $T \sim 1.95°K$. Since the scaling parameter η is extremely large, $\eta = T_D/T \ \sim 1MeV/1.9°K \sim 10^{10}$, the effective neutrino mass m_{eff} is negligibly small, and we can safely take $m_{eff} = 0$. Therefore the present momentum spectrum $F(p)$ is well approximated by the function of the thermal Fermi-Dirac distribution for relativistic massless paticles,

$$F(p) = 1/(e^{(p-\mu)/T}+1) \tag{4}$$

There are three kinds of neutrinos with different flavor (electron-neutrino, mu-neutrino, and tau-neutrino) and they can be either Dirac or Majorana neutrinos. Hereafter we shall assume that all neutrinos are Dirac neutrinos unless otherwise stated. In the case of Dirac neutrinos, neutrinos and antineutrinos are different particles and the chemical potentials of neutrtino and antineutrino are equal in magnitude but opposite in sign. The number density asymmetry for neutrinos due to the opposite sign of chemical potential is analytically given by

$$\rho_{\nu_\alpha} - \rho_{\bar{\nu}_\alpha} = (T^3/6)(\xi_\alpha + \xi_\alpha{}^3/\pi^2) \quad \text{with} \quad \xi_\alpha = \mu_{\nu_\alpha}/T \tag{5}$$

The equation (5) shows $\rho_{\nu_\alpha} > \rho_{\bar{\nu}_\alpha}$ for $\xi_\alpha > 0$ and $\rho_{\nu_\alpha} < \rho_{\bar{\nu}_\alpha}$ for $\xi_\alpha < 0$. The analysis of the Big Bang Nucleosynthesis[6] gives strong constraints on the chemical potential of electron neutrino, of the order of $|\xi_{\nu_e}| < 0.1$. The above arguments are affected by the neutrino oscillation or resonant MSW oscillation during the expansion of the universe. The LMA solution for the solar neutrino oscillation confirmed by Kamland experiment together with the large mixing found in the atmospheric neutrino oscillation leads to the strong coupling among the chemical potentials of all the three neutrinos during the universe expansion, and the standard arguments suggest that the strong coupling leads to a nearly equal value of all the chemical potentials before the period of Big Bang Nucleosynthesis. This gives rather severe constraints[6] of the order of $|\xi_\alpha| < 0.1$ for all the neutrinos ν_e, ν_μ and ν_τ. We hereafter assume that the chemical potential is negligibly small ($|\xi| << 1$) as usual, although possibility of existence of a larger lepton asymmetry cannot be excluded yet.

If the relic neutrinos are massive and are at present nonrelativistic, $T << m_\nu$, they may have been scattered or trapped by the gravitational force of galaxies or super galaxies, and have lost the memory of the helicity which used to be –1 or +1 for neutrinos or antineutrinos, respectively, in the early period of high temperature.

The neutrinos trapped by galaxies or super galaxies may acquire momentum larger than T if the neutrino mass is large, and the neutrino phase volume may become larger, $p^3 > T^3$, and the neutrino density can be enhanced by gravitational clustering. The gravitational clustering effect is larger for heavier neutrinos, and a simulation[7] has given the density enhancement factor in our Milky Way of about 20 for $m_\nu = 0.60eV$ and 1.4 for $0.15eV$. The enhancement factor depends only on the mass and is equal to neutrino and antineutrino with the same mass. In the following, however, we neglect such gravitational enhancement, since we have not enough knowledge of the mass values of neutrinos.

A neutrino flavor eigenstate is a mixture of three different mass eigenstates, and the flavor of neutrino changes in time as it propagates through space. Various neutrino oscillation experiments have provided data on the mass differences between the different mass eigenstates of neutrino. Defining $\Delta m^2_{ij} \equiv m_i^2 - m_j^2$, they are[8]

(i) $\Delta m_{21}{}^2 = 7.46 \sim 7.89 \times 10^{-5} eV^2$

(iia) $\Delta m_{31}{}^2 = 2.04 \sim 2.93 \times 10^{-3} eV^2$ (6)

(iib) $\Delta m_{13}{}^2 = 1.94 \sim 2.83 \times 10^{-3} eV^2$

Here the mass squared difference (i) has been determined by the solar and reactor neutrino experiments, and is mainly responsible for $\nu_e \leftrightarrow \nu_\mu$ or ν_τ. The mass squared difference (iia) has been determined by the experiments with atmospheric neutrinos and K2K+Minos accelerator neutrinos in the case of normal mass hierarchy ($m_1 < m_2 << m_3$), and is mainly responsible for $\nu_\mu \leftrightarrow \nu_\tau$, while the mass squared difference (iib) is the same as (iia) but in the case of inverted mass hierarchy ($m_3 << m_1 < m_2$).

Although we cannot uniquely determine the masses and the mass hierarchy of neutrinos with these experimental results only, we see that at least the most and the second massive neutrinos have masses larger than $0.05eV$ and $0.009eV$, respectively., It is likely for them to have masses of the order of $10^{-1} \sim 10^{-2} eV$, while the lightest neutrino could be a massless particle. From the momentum distribution (4) of relic neutrinos the magnitude of typical momentum of relic neutrinos is $p \sim T \sim 1.95°K$, $\sim 1.6 \times 10^{-4} eV$ which is much smaller than the lower limit of the mass range of the heaviest two neutrinos. Thus at least the two heavier relic neutrinos can be treated as nonrelativistic particles.

For the typical momentum $p \sim 1.6 \times 10^{-4} eV$, the neutrino wavelength λ is $\lambda = h / p \sim 0.77cm$, which is much larger than the inter-atomic spacing of most materials. Because of this long wavelength the neutrino interaction with matter via the weak interaction would be predominantly coherent involving the recoil of the whole target material rather than individual atoms. As is the familiar case of slow neutron motion in matter, the relic neutrinos would respond to an average interaction potential in matter, which corresponds to a neutrino refractive index differing from unity. Since the rate of absorption of low energy neutrino by matter is negligibly small, the refraction index of matter is approximately real. Thus neutrinos would undergo reflection and refraction at material surfaces with associated momentum transfers to the material.

Although the refractive index of neutrino in matter had been already estimated in various papers[4] we shall evaluate in the next two sections the refractive index n of neutrinos in matter for the sake of completeness, and study the possibility of total reflection of neutrinos at material surfaces, thereby taking into account the estimated masses of three kinds of neutrino and the flavor mixing due to neutrino oscillations.

2. Refractive index of relic neutrinos in matter and neutrino potential

2-1 Relation between refractive index and the potential

The nuclei and electrons in a solid are tightly bound together so that small energy transfers to such materials by collisions of relic neutrinos are not enough to excite the matter, and only elastic scatterings occur in relic neutrino-matter collisions. Furthermore wavelengths of relic neutrinos are much larger than the inter-atomic spacing so that neutrino scattering amplitudes off many atoms of the material add up coherently to amplify the scattering amplitude. Because of the large wavelength of a relic neutrino and a small energy transfer in neutrino collision with matter we can usually describe neutrino-matter scatterings in terms of refractive index of relic neutrinos in matter similar to the case of slow neutron scatterings in matter.

The index of refraction is defined as the ratio of the momenta,

$$n = p'/p, \tag{7}$$

where p and p' are the momenta of the relic neutrino in the vacuum and that in the matter, respectively. The momentum in the vacuum p is determined with the energy $E = \sqrt{p^2 + m^2}$, and the momentum in matter p' is related to the weak potential energy U made between a relic neutrino and atoms in the medium.

(i) Massive neutrinos in the nonrelativistic limit:

In the nonrelativistic case with $p, |U| << m$ the relation of the momentum to energy is simply

$$E - m = p^2/2m = p'^2/2m + U \tag{8}$$

which gives the refractive index n as

$$n^2 = (p'/p)^2 = 1 - 2mU/p^2. \tag{8'}$$

For a small potential $|U| \ll p^2/2m$ or $|n-1| << 1$, Eq.(8) approximately gives

$$n - 1 = -mU/p^2 \tag{8''}$$

The refractive index n is also related to the forward scattering amplitude $f(0)$ of neutrino in matter:

$$n - 1 = (2\pi/p^2) f(0) . \tag{9}$$

from which we have

$$f(0) = -mU/2\pi . \tag{9'}$$

(ii) Massless neutrinos (or extremely relativistic neutrinos)

The momentum in matter is given by the energy conservation

$$p' = E - U = p - U , \tag{10}$$

for $E > U$, and the refractive index n is given by

$$n = 1 - U / p\,. \tag{11}$$

In the case of $E < U$ a solution

$$p' = U - p$$

is possible, and then the refraction index is given

by $n = U / p - 1$. (11')

A special care will be given for this case in the Appendix A.

(iii) Massive neutrinos in general:

In general the weak potential in materials without spin polarization has a matrix dependence (see the next section for detail), and it is expressed as

$$U = V(1-\gamma^5)\,. \tag{12}$$

The matrix dependence disappears only for the nonrelativistic limit ($<\gamma^5> = 0$),

$$U = V \tag{13}$$

and for the massless or extremely relativistic limit ($<\gamma^5> = -1$)

$$U = 2V\,. \tag{14}$$

In general the energy–momentum relation for neutrinos in the weak potential U (Eq.(12)) is given by

$$(E-V)^2 = m^2 + (p'h' - V)^2 \quad , \tag{15}$$

where h' is the neutrino helicity in the matter. (The derivation is given in the following Section 2-2-1). For real p' we have $(E-V)^2 - m^2 > 0$ or equivalently $1 - 2EV / p^2 + V^2/p^2 > 0$, and we obtain the refractive index n from Eqs.(7) and (15) as

$$n = p' / p = h'V / p + \sqrt{1 - 2EV / p^2 + V^2 / p^2} \qquad (h' = \pm 1) \tag{16a}$$

for $p' > h'V$ (and $1 - 2EV/p^2 < 0$ if $h'V < 0$), or

$$n = p' / p = h'V / p - \sqrt{1 - 2EV / p^2 + V^2 / p^2} \tag{16b}$$

for $p' < h'V$ and $1 - 2EV/p^2 < 0$. The latter case is possible only for $V > 0$ and $h' = 1$. We note that in the case of antineutrino (see Section 2-2-1 for the derivation) Eq.(15) is modified to the following

$$(E+V)^2 = m^2 + (p'h' - V)^2$$

which gives

$$n = p' / p = h'V / p \pm \sqrt{1 + 2EV / p^2 + V^2 / p^2} \quad . \tag{16c}$$

and in the case of Majorana neutrino (see Eq.(30M) in the next Section 2-2-1)

$$E^2 = m^2 + (p'h' - 2V)^2 ,$$

which leads to

$$1=(n-2h'V/p)^2 \quad . \qquad (16d)$$

2-2. The neutrino potential in matter

2-2-1 Neutrino potential for flavor eigenstates

Although it is appropriate to treat the reflection or refraction of relic neutrinos with mass eigenstates rather than flavor eigenstates, we give here the neutrino potential for flavor eigenstates, at first; this is because the weak interaction is diagonal and easy to treat with for the flavor eigenstates. The weak potential for the relic neutrinos in matter is made both by the neutral current interaction of them with nucleons and electrons in matter and the charged current interaction of the electron-neutrino ν_e with electrons in matter. The part of the Hamiltonian density for a neutrino and a nucleon due to the weak neutral current interaction is given by,

$$H_N=(G_F/\sqrt{2})\bar{\nu}\gamma^{\mu}(1-\gamma^5)\nu\cdot\bar{N}\gamma_{\mu}\{(g_V+g_A\gamma^5)(\tau^3/2)-2Qg_V\sin^2\theta_W)\}N\ , \qquad (17)$$

where ν and $\bar{\nu}$ are the neutrino fields, N and $\bar{N}$ are neucleon fields. Here Q is the charge operator of nucleon ($Q=1$ for proton, and $Q=0$ for neutron), $\tau^3/2$ is the weak isospin operator of nucleon ($\tau^3/2=1/2$ for proton and $\tau^3/2=-1/2$ for neutron), θ_W is the Weinberg angle ($\sin^2\theta_W\approx 0.226$), and $g_V=1$, $g_A=-1.26$. The expectation value of H_N of Eq.(17) in matter gives the potential energy U_N due to the weak neutral current interaction of the neutrino with nucleons in matter. Since we can treat nucleons in the extremely nonrelativistic limit, we obtain

$$\begin{aligned}&<\bar{N}\gamma_{\mu}N>=\delta_{\mu 0}\rho_N\\ &<\bar{N}\gamma_i\gamma^5 N>=-<\bar{N}\gamma^i\gamma^5 N>=-<\sigma_N{}^i>\rho_N \quad (i=1,2,3)\\ &<\bar{N}\gamma_0\gamma^5 N>=0\end{aligned} \qquad (18)$$

where $<\sigma_N{}^i>$ is the average of the nucleon polarization of nuclei in matter. The expectation value of the weak current for the neutrino is defined as

$$\begin{aligned}&<\int d^3x\cdot\bar{\nu}\gamma^0(1-\gamma^5)\nu>=2K\\ &<\int d^3x\cdot\bar{\nu}\gamma^i(1-\gamma^5)\nu>=2K^i \qquad (i=1,2,3)\end{aligned} \qquad (19)$$

Here the function K is given by

$$K=(N_\nu-v'h')/2 \qquad (N_\nu=\pm 1) \qquad (20D)$$

with v' the velocity of the neutrino in matter and h' its helicity. We have $N_\nu=1$ or -1 for neutrino and antineutrino, respectively. The function K^i is given by

$$K^i=(p'^i/p')(N_\nu v'-h')/2=N_\nu v'(p'^i/p')/2-<\sigma_\nu{}^i>/2 \qquad (21D)$$

with $\vec{p}'$ the momentum of the neutrino in matter and $<\sigma_\nu{}^i>$ its spin polarization.
For extreme cases we have

$$K = 1/2 \ \text{ or } \ -1/2 \tag{22a}$$

for extremely nonrelativistic Dirac neutrino or antineutrino, respectively, and

$$K = 1 \ \text{ or } \ -1 \tag{22b}$$

for massless neutrino or antineutrino, respectively.
In case of Majorana neutrino (see Appendix B), we have

$$K = -v'h' \ . \tag{20M}$$

$$K^i = -(p'^i / p')h' = -<\sigma_\nu{}^i> \tag{21M}$$

In case of extremely nonrelativsitic Majorana neutrino we have $K = 0$ and in the relativsitic limit we have $K = -h' = \pm 1$. For most materials the average nucleon polarization $<\sigma_N{}^i>$ is zero, and the potential energy U^N for neutrino due to the neutrino-nucleus weak interaction in matter is given by

$$U^N = 2KV^N, \quad \text{with} \quad V^N = (G_F/\sqrt{2})\{(1/2 - 2\sin^2\theta_W)\rho_p - (1/2)\rho_n\} \tag{23}$$

where ρ_p and ρ_n are the number density of proton and that of neutron in matter, respectively.

The part of the potential energy of a neutrino U^e due to its weak interaction with electrons is calculated similarly and is given as

$$U^e = 2KV^e$$

with

$$V^e = (G_F/\sqrt{2})(1/2 + 2\sin^2\theta_W)\rho_e \qquad \text{for } \nu_e \ , \tag{24a}$$

$$\text{or} \qquad = (G_F/\sqrt{2})(-1/2 + 2\sin^2\theta_W)\rho_e \qquad \text{for } \nu_\mu, \nu_\tau \ . \tag{24b}$$

Here ρ_e is the number density of electrons in matter. Both the weak charged current interaction and the neutral current interaction are taken into account. We have assumed that electrons in matter are nonrelativistic particles and their average spin polarization is zero. Since ordinary materials with neutral charge have $\rho_p = \rho_e$, the total potential U which is the sum of U^N and U^e is given by

$$U = 2KV \tag{25}$$

with

$$V = (G_F/2\sqrt{2})(2\rho_p - \rho_n) \qquad \text{for } \nu_e \tag{26a}$$

or $\qquad = -(G_F/2\sqrt{2})\rho_n \qquad$ for ν_μ, ν_τ (26b)

Terms in U^N and U^e proportional to $\sin^2\theta_W$ are cancelled to each other. Using the potential Eq.(12), we can express the effective Hamiltonian for the neutrino as,

$$H = \vec{\alpha}\vec{p}' + m\beta + U, \qquad U \equiv V(1\text{-}\gamma^5)\ . \tag{27}$$

Here $\vec{p}'$ is the momentum in matter and $\alpha_i = \gamma^0\gamma^i$, $\beta = \gamma^0$ are Dirac matrices given below. We note that the potential of the Hamiltonian in Eq.(27) should be changed as

$$V(1-\gamma^5) \to V(-1-\gamma^5), \tag{27'}$$

for the antineutrino, and as

$$V(1-\gamma^5) \to -2V\gamma^5 \tag{27''}$$

for the Majorana neutrino (see Appendix B).

We use the convention of Dirac matrices as follows,

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\ , \text{ and } \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{28}$$

Here σ^i's are 2×2 Pauli spin matrices and I is 2×2 identity matrix.

If we use the helicity eigenstate in matter with helicity h' ($\vec{\sigma}\vec{p}' = p'h'$, $h' = \pm 1$), we can simplify the Dirac equation for the energy eigenstate ψ with energy eigenvalue E as below:

$$(E-V)\psi = (p'h' - V)\gamma^5\psi + m\beta\psi\ . \tag{29}$$

We see that Eq.(29) is also obtained by the substitutions $E \to E - V$ and $p \to p' - h'V$. in the free Dirac equation. Therefore the energy momentum relation in matter is given by these substitutions in the free relation $E^2 = m^2 + p^2$ as

$$(E-V)^2 = m^2 + (p' - h'V)^2\ \ , \tag{30D}$$

for the neutrino and,

$$(E+V)^2 = m^2 + (p' - h'V)^2\ \ , \tag{30D'}$$

for the antineutrino.

In case of Majorana neutrino (see Appendix B), the potential term in Eq.(27) should be modified as $V(1-\gamma^5) \to -2V\gamma^5$, which modifies the substitution rule of (A6) as $E \to E$ and $p \to p' - 2h'V$ to obtain the energy momentum relation in matter from the free relation $E^2 = m^2 + p^2$,

$$E^2 = m^2 + (p' - 2h'V)^2 \tag{30M}$$

The equation (30D) may allow not only a solution $E = \sqrt{m^2 + (p' - h'V)^2} + V$ but also a solution with the negative sign in front of the square root,

$$E = -\sqrt{m^2 + (p' - h'V)^2} + V \,, \qquad (31)$$

since the right hand side of Eq.(31) can be positive for a very large positive potential with $V > E + m$. Such a case will be discussed in the Section 3 and Appendix A. When electrons in matter are polarized with an average spin $<\vec{\sigma}_e>$, the potential for nonrelativistic neutrinos is modified as follows:

$$\begin{aligned} U \;&=\; (G_F / 2\sqrt{2})\{2K(2\rho_p - \rho_n) + 2\vec{K}\rho_e < \vec{\sigma}_e >\} \qquad \text{for } \nu_e \\ \text{or} \;&=\; (G_F / 2\sqrt{2})(-2K\rho_n - 2\vec{K}\rho_e < \vec{\sigma}_e >) \qquad \text{for } \nu_\nu, \nu_\tau \end{aligned} \qquad (32)$$

Here the coefficients K and $\vec{K}$ are given in Eqs.(20),(21) and (22). For example, we recall $K = 1/2$ and $\vec{K} = -<\vec{\sigma}_\nu>/2$ for nonrelativistic Dirac neutrinos.

As a typical example let us consider the case of iron as a material. If we denote the number density of iron atoms by ρ_{Fe}, which is $\sim 0.85 \times 10^{23}$ atoms/cm^3, we have $\rho_p = \rho_e = 26\rho_{Fe}$, and $\rho_n = 30\rho_{Fe}$. Then from Eq.(32) the potential U for nonrelativistic massive neutrinos is given by

$$\begin{aligned} U &= (G_F / 2\sqrt{2})22\rho_{Fe} \;\sim\; 5.7 \times 10^{-14} eV \qquad \textit{for } \nu_e \\ U &= -(G_F / 2\sqrt{2})30\rho_{Fe} \sim -7.9 \times 10^{-14} eV \qquad \textit{for } \nu_\mu, \nu_\tau \end{aligned} \qquad (33)$$

For massless or extremely light ($p >> m$) neutrinos, the value of U must be multiplied by a factor 2, and for antineutrino the value of U changes the above sign due to the factor K given in Eqs.(20),(21) and (22).

The refractive index n for massive neutrinos in general is obtained by putting $p' = pn$ in Eq.(30D),

$$(n - h'V/p)^2 = [(E - V)^2 - m^2]/p^2 \qquad (33D)$$

which reduces to Eq.(8') for the nonrelativistivc limit $p, |V| << m$ with $U = V$, and to Eq.(11) for massless case with $U = V(1 - h') < E$, and Eq.(11') for massless case with $U = V(1 - h') > E$. The latter case will be discuused in Appendix A.

For Majorana neutrino case the refractive index is given by setting $p' = pn$ in Eq.(30M),

$$n = 1 + 2h'V/p \qquad (33M)$$

2-2-2 Neutrino oscillation and effective neutrino potential for mass eigenstates

The weak potential for neutrinos in matter is diagonal for the flavor of neutrino. However, the neutrino flavor changes due to neutrino oscillation during the propagation of neutrino waves through space, and also due to flavor-changing reflection or transmission when they are reflected or transmitted at boundary surfaces of matter. Although the oscillation length L for high energy neutrinos is large,

$$L \approx 2E / \Delta m^2 \approx 160\mathrm{m}\left(E / \mathrm{MeV}\right)\left(2.5\times 10^{-3}\mathrm{eV}^2 / \Delta m^2\right)$$

where E is the neutrino energy and Δm^2 is the mass squared difference between the two mass eigenvalues, the length L for relic neutrinos is much shorter and is less than an order of $1/|\Delta m|$,

$$L < 1/ \,/\, \Delta m \,/ < 1\Big/\sqrt{\Delta m_{21}{}^2} \sim 2\times 10^{-3}\mathrm{cm}\,, \tag{34}$$

where the mass squared difference Eq,(6) is used. Since the oscillation length is small and the neutrino flavor changes rapidly, it is more appropriate to use the neutrino mass eigenstates instead of the flavor eigenstates to describe relic neutrinos. Furthermore as seen from the comparison of Eq.(33) with Eq.(6), the potential U for relic neutrinos in matter is much smaller than mass differences among neutrinos in different mass eigenstates, so that each neutrino mass eigenstate is not much disturbed when it is reflected at boundary surfaces of matter (see Appendix D for the details). Therefore it is a quite good approximation to take only the diagonal part of the weak potential in mass eigenstate representation, when we consider the reflection and the refraction of relic neutrinos on the surfaces of materials. We estimate the weak potential for mass eigenstates in the following.

Let us denote the mass eigenstate as $|\nu_i\rangle$ $(i = 1,2,3)$ and the flavor eigenstates as $|\nu_\alpha\rangle$ $(\alpha = e,\mu,\tau)$. The relation between the mass and flavor eigenstates is expressed by the 3×3 MNS (Maki, Nakagawa and Sakata) unitary matrix $W = (W_{\alpha i})$ as

$$|\nu_\alpha\rangle = \sum_{i=1,2,3} W_{\alpha i}|\nu_i\rangle\,. \tag{35}$$

For example the electron-neutrino state is expressed as the following mixture of three mass eigenstates,

$$|\nu_e\rangle = \sum_{i=1,2,3} W_{ei}|\nu_i\rangle, \qquad \Big(\sum_{i=1,2,3}|W_{ei}|^2 = 1\Big) \tag{36}$$

Now with the equations (26a) and (26b) the potential U for neutrinos in unpolarized matter can be expressed as

$$U = 2KV\,, \quad \text{with} \ \ V = (G_F / 2\sqrt{2})(-\rho_n + 2\rho_p P_e)\,, \tag{37}$$

where P_e is the projection operator upon the electron-neutrino state and is given by

$$P_e = |\nu_e\rangle\langle\nu_e| = W_{ei}W_{ej}^* |\nu_i\rangle\langle\nu_j| . \tag{38}$$

Then U becomes a 3×3 matrix in mass eigenstate representation given by

$$U = 2K(G_F/2\sqrt{2})\times\left(-\rho_n + 2\rho_p\begin{pmatrix} |W_{e1}|^2 & W_{e1}W_{e2}^* & W_{e1}W_{e3}^* \\ W_{e2}W_{e1}^* & |W_{e2}|^2 & W_{e2}W_{e3}^* \\ W_{e3}W_{e1}^* & W_{e3}W_{e2}^* & |W_{e3}|^2 \end{pmatrix}\right) \tag{39}$$

When we consider the reflection from the surface of materials, we take into account the off-diagonal elements of the potential given by Eq.(39). The off-diagonal elements can transform a mass eigenstate into another when a neutrino is reflected or refracted at boundary surfaces of a material. Such a process is possible only when the mass of the incident neutrino ν_i is heavier than that of the reflected one ν_f, namely, $m_i > m_f$, enabling exoergic reflection, since the momentum p of relic neutrino is much less than the mass difference and the reflection for $m_f > m_i$ is energetically impossible. However, the probability for such inelastic reflection or refraction to occur transforming a mass eigenstate ν_i to another ν_f is found to be very small by a factor of the order of $(p/m_i)^3 \sim (T/m_i)^3 < 10^{-5}$, compared with the elastic reflection (see Appendix D). Furthermore the elastic reflection has a large enhancement for small grazing angle, while the inelastic one has no such enhancement. Especially when the grazing angle is small enough to make the total reflection, the inelastic reflection is relatively negligible. Although such a mass changing reflection has a large recoil momentum $\sim \sqrt{m_i^2 - m_f^2}$ rather than $\sim p$ of the elastic reflection, the smallness of the rate of the inelastic reflection gives the net effective rate of momentum transfer smaller by a factor of $(p/m_i)^2 \sim (T/m_i)^2 < 10^{-3}$ compared with the elastic reflection. Therefore we can safely neglect this off-diagonal effect even if neutrinos are reflected or transmitted many times at boundary surfaces of materials. We have thus effective neutrino potential which is well approximated by taking only diagonal elements of Eq.(39), and is given for each mass eigenstate ν_i:

$$U = 2K(G_F/2\sqrt{2})(-\rho_n + 2\rho_p|W_{ei}|^2) \tag{40}$$

The MNS matrix, (here the complex phase is neglected),

$$W = \begin{pmatrix} W_{e1} & W_{e2} & W_{e3} \\ W_{\mu 1} & W_{\mu 2} & W_{\mu 3} \\ W_{\tau 1} & W_{\tau 2} & W_{\tau 3} \end{pmatrix} = \begin{pmatrix} c_{13}c_{12} & s_{12}c_{13} & \pm s_{13} \\ -s_{12}c_{23} & c_{23}c_{12} \mp s_{23}s_{13}s_{12} & s_{23}c_{13} \\ s_{23}s_{12} \mp s_{13}c_{23}c_{12} & -s_{23}c_{12} \mp s_{13}s_{12}c_{23} & c_{23}c_{13} \end{pmatrix}, \tag{41}$$

with the abreviations, $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$, is obtained by the analysis of the present experimental data[9]

$$\begin{aligned} \sin^2\theta_{13} &= 0.009 \times (1^{+0.023}_{-0.009}) \\ \sin^2\theta_{12} &= 0.314 \times (1^{+0.18}_{-0.15}) \\ \sin^2\theta_{23} &= 0.44 \times (1^{+0.18}_{-0.15}) \end{aligned} \tag{42}$$

The double sign in Eq.(41) corresponds to the CP conserving phase $e^{i\delta} = \pm 1$. If we take the central values for $\sin^2\theta_{ij}$, we have an approximate matrix,

$$W \sim \begin{pmatrix} 0.824 & 0.557 & \pm 0.094 \\ -0.419 \mp 0.062 & 0.619 \mp 0.035 & 0.660 \\ 0.371 \mp 0.058 & -0.549 \mp 0.039 & 0.744 \end{pmatrix} \tag{43}$$

with the double sign corresponding to $e^{i\delta} = \pm 1$.

Although we do not know precise values of $W_{\alpha i}$ and particularly that of W_{e3}, let us use, for simplified discussions, the following approximate values which are consistent with the present experimental values given in Eq.(42):

$$\begin{aligned} &W_{e1} = \sqrt{2/3}, \quad W_{\mu 1} = -W_{\tau 1} = -\sqrt{1/6}, \quad W_{e2} = W_{\mu 2} = -W_{\tau 2} = \sqrt{1/3} \\ &W_{e3} = 0, \quad W_{\mu 3} = W_{\tau 3} = \sqrt{1/2} \end{aligned} \tag{44}$$

Then the flavor eigenstates become the following linear combinations of the mass eigenstates:

$$\begin{aligned} |\nu_e\rangle &= \sqrt{2/3}|\nu_1\rangle + \sqrt{1/3}|\nu_2\rangle \\ |\nu_\mu\rangle &= -\sqrt{1/2}\left(\sqrt{1/3}|\nu_1\rangle - \sqrt{2/3}|\nu_2\rangle\right) + \sqrt{1/2}|\nu_3\rangle \\ |\nu_\tau\rangle &= \sqrt{1/2}\left(\sqrt{1/3}|\nu_1\rangle - \sqrt{2/3}|\nu_2\rangle\right) + \sqrt{1/2}|\nu_3\rangle \end{aligned}$$

If we use the values of $W_{\alpha i}$ given by the equations (44), we have the potential for relic neutrinos in iron, for examlple, as follows:

$$\begin{aligned} U &= 2K(G_F\rho_{Fe}/2\sqrt{2}) \times \left(-30 \times I + 2 \times 26 \times \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \\ &\approx 2K \cdot \begin{pmatrix} 1.2 \times 10^{-14} eV & 0 & 0 \\ 0 & -3.3 \times 10^{-14} eV & 0 \\ 0 & 0 & -7.8 \times 10^{-14} eV \end{pmatrix}, \end{aligned} \tag{39'}$$

with ρ_{Fe} the number density of iron atoms. We have omitted the off-diagonal elements to obtain Eq.(39') as is discussed above. The sign of U is reversed for antineutrinos.

For most materials including iron the potential U for neutrinos in matter can be regarded diagonal, and U is positive only for the mass eigenstate ν_1, $\bar{\nu}_2$ or $\bar{\nu}_3$. Thus the total reflection of massive neutrinos at surfaces of matter only occurs for neutrino ν_1 and antineutrinos $\bar{\nu}_2$ and $\bar{\nu}_3$, when appropriate conditions such as the kinetic energy $\varepsilon < U$ and mass $2m_i > U$.

At least two of the three kinds of neutrino have a mass of the order of $10^{-1}eV \sim 10^{-2}eV$ which is much larger than the typical relic neutrino momentum $p = kT \sim 1.6\times10^{-4}eV$. These massive neutrinos can be treated as nonrelativistic particles, and Eqs. (8") and (39') give the refractive index of iron for each mass eigenstate of relic neutrino with mass m_i and momentum p as

$$\begin{aligned} n_1 - 1 &\sim -0.5\times10^{-7}(m_1/10^{-1}eV)(1.6\times10^{-4}eV/p)^2 \quad \text{for } \nu_1 \\ n_2 - 1 &\sim +1.3\times10^{-7}(m_2/10^{-1}eV)(1.6\times10^{-4}eV/p)^2 \quad \text{for } \nu_2 \\ n_3 - 1 &\sim +3.1\times10^{-7}(m_3/10^{-1}eV)(1.6\times10^{-4}eV/p)^2 \quad \text{for } \nu_3 \end{aligned} \tag{45}$$

It should, however, be noted that the above nonrelativistic approximation may not be applicable for the lightest neutrino (ν_1 for the normal hierarchy case and ν_3 for the inverted hierarchy case), since it can be very light or even massless. If one of the three kinds of neutrinos is massless, we use Eqs.(11) and (39') to obtain the refraction index. In doing so we multiply the potential used in deriving Eq.(45) by 2, since we have $K=1$ instead of 1/2 for massless neutrinos. Then for the massless neutrino with momentum we obtain

$$n_1 - 1 \sim -1.5\times10^{-10}(1.6\times10^{-4}eV/p) \tag{46a}$$

if ν_1 is massless (normal hierarchy), and

$$n_3 - 1 \sim +1.0\times10^{-9}(1.6\times10^{-4}eV/p) \tag{46b}$$

if ν_3 is massless (inverted hierarchy).

3. Total reflection of neutrinos at boundary surfaces of a material

Next we shall consider total reflection of neutrinos at boundary surfaces of a material. From the equation (8') the refractive index n for massive neutrinos becomes imaginary ($n^2 < 0$) for ultra low energy relic neutrinos with kinetic energy $\varepsilon < U$. Then the neutrino momentum p' ($= np$) in matter becomes imaginary and the neutrino wave decays exponentially in matter so that it cannot propagate deep inside the matter. Therefore those ultra low energy relic neutrinos are totally reflected at the boundary surfaces of the material irrespective of its impinging angles at the boundary surface. The positive potential $U > 0$ implies a repulsive potential for neutrinos and the total reflection (independent of the angle of incidence) occurs for neutrinos with kinetic energy $\varepsilon < U$. The fraction f of those totally reflected ultra low energy relic neutrinos is extremely small, only of the order of $f \sim (\sqrt{2mU}/T)^3$. (In a single reflection, however, a fraction of $f \sim \sqrt{2mU}/T$ of neutrinos wihich have small enough grazing angle are totally reflected). If we assume $U \sim +7.8\times10^{-14}eV$ and $T = 1.6\times10^{-4}eV$, we obtain

$$f \sim (\sqrt{2mU}/T)^3 \sim 1.6\times10^{-10}(m/0.05eV)^{3/2}.$$

On the other hand a massless neutrino with energy $E < U$ is not necessarily totally reflected at the boundary and may pass through the material without being reflected. This is because the neutrino with $E = p$ in the vacuum take the energy level in matter with $E = U - p'$ if $E < U$ holds. This energy level corresponds to the negative energy level in the vacuum, but it is positive owing to the repulsive potential. More peculiar phenomena take place when the massless neutrinos hit the wall with the direction perpendicular to the surface. In this case the neutrinos are not reflected at all, and they all penetrate through the wall (see Appendix A) by 100%. The neutrino reflection is forbidden in this case. This is understood by considering the perfect conservation of both the angular momentum and the chirality at the same time: let us define the direction of the incident neutrino momentum as the z-direction and assume the z-axis is perpendicular to the surface plane. When the neutrino vertically hits the surface of the matter, the z-component of the total angular momentum is conserved, and has no contribution from the orbital angular momentum. It consists only of of the spin angular momentum. Therefore the z-direction of spin should be conserved in the vertical hit. The massless neutrino conserves the chirality, and it is equal to the helicity when the neutrino is in the vacuum. If the neutrino is reflected at the surface, the z-component of the spin is conserved while the momentum direction is reversed. Then the helicity should be reversed, and it violates the conservation of chirality. Thus it is impossible for the reflection to conserve the helicity and z-component of the spin at the same time. This forbids the reflection of the neutrino hitting vertically the surface. It is only when the grazing angle is less than a critical value that the total reflection of neutrino is realized (see the Appendix A).

Total reflection of neutrinos also occurs for massive neutrinos with kinetic energy $\varepsilon > U$ if neutrinos impinge upon a material almost tangentially to its boundary surface. Let us take the boundary surface as x-y plane and z-axis perpendicular to the surface. When a neutrino impinges upon the material boundary with a grazing angle θ to the surface plane (see Fig.1), the boundary conditions at the surface are

$$p' = np \tag{47}$$

$$p'_x = p_x, \quad p'_y = p_y \tag{48}$$

where p and p' are the neutrino momentum in the vacuum and that in the material, respectively. We note that the momentum parallel to the surface is conserved because the system is invariant under the infinitesimal parallel displacement along the boundary surface, and the Hamiltonian H for the neutrino satisfies $\partial H/\partial x = \partial H/\partial y = 0$. This means that the momentum operators P_x and P_y commutes with H, which leads to the conservation of P_x and P_y. From Eqs.(47) and (48) we have

$$p\cos\theta = p'\cos\theta' = np\cos\theta' \quad \therefore \cos\theta = n\cos\theta',$$

which is known as Snell's law in optics. We have also

$$p_z'^2 = (np)^2 - p_x{}^2 - p_y{}^2 = p^2(n^2 - \cos^2\theta) \tag{49}$$

Thus for grazing angle θ less than a critical angle θ_C given by $\cos\theta_C = n$ ($n<1$) p'_z becomes imaginary and neutrinos cannot penetrate deep into the material and total reflection occurs. Since $n-1$ is very small, the critical angle is also small and is given by

$$\theta_C = \sin^{-1}\sqrt{1-n^2} \approx \sqrt{2(1-n)} \tag{50}$$

Only those neutrinos impinging almost tangentially to the boundary surface with a grazing angle θ less than θ_C are totally reflected.

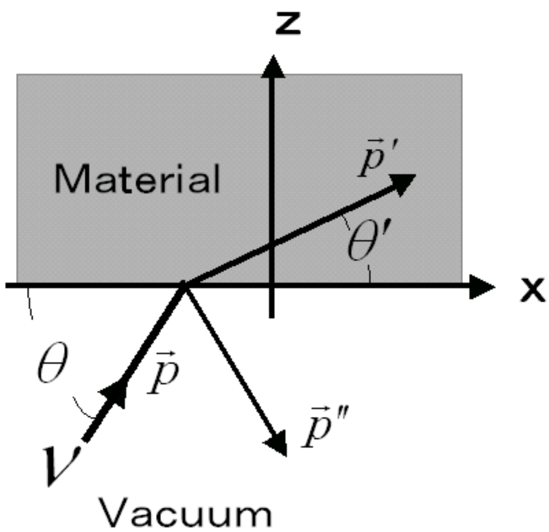

Fig.1 Grazing angle θ

For a small $n-1$ and θ_C we obtain from Eq.(50),

$$p_z'^2 = p^2(\cos^2\theta_C - \cos^2\theta) \approx p^2(\theta^2 - \theta_C{}^2) \tag{51}$$

Neutrinos satisfying the condition for the total reflection at a bounday surface of material can still enter into the material to a certain depth. Since the neutrino wave amplitude decreases exponentially as

$$\exp[ip'_z z] \sim \exp[-p\sqrt{\theta_C{}^2 - \theta^2}\, z] = \exp[-z/z_0] \tag{52}$$

inside the matter, the decay length z_0 of the neutrino wave in matter is given by

$$z_0 = p^{-1}(\theta_C{}^2 - \theta^2)^{-1/2} > p^{-1}\theta_C{}^{-1} = 1/\sqrt{2mU} \qquad (53)$$

for massive relic neutrinos. This lower limit of z_0 is independent of neutrino momentum p. For example if we use $U = 7.8\times10^{-14}eV$ of iron for antineutrino $\bar{\nu}_3$, which is the heaviest in case of the normal hierarchy, we obtain

$$z_0 \geq 1/\sqrt{2mU} = 222\text{cm}\times(0.05eV/m_{\nu_3})^{1/2}.$$

As was already pointed out by Langacker et.al.[10], the total reflection of neutrinos only occurs if the thickness of material is much larger than the neutrino decay length z_0. Otherwise the neutrino wave amplitude is still substantial at the opposite side of the material implying a large probability for neutrinos passing through the material. Since z_0 is of the order of several meters, rather thick materials are needed for total reflection of neutrinos to occur.

According to the standard Big Bang model, relic neutrinos were mostly left-handedand and relic antineutrinos were mostly right-handed, and they may keep their handedness during the expansion of the universe if the momentum is much larger than its mass. When neutrinos become nonrelativistic, the Hamiltonian for neutrino in gravitational potential[11] does not commute with the helicity operator $\vec{\sigma}\vec{p}/p$, but commutes with the spin operator $\vec{\sigma}$. Thus the neutrino does not conserve the helicity but rather keeps its spin direction during the motion under the gravitaional potential made by stars, galaxies or superclusters and so on. If the speed of neutrino $\beta = v/c$ is considerably smaller than 10^{-3}, the typical order of the rotational velocities of stars in Our Galaxy or the peculiar velocities of galaxies in the Universe, they may lose their memory of helicity during its motion under the gravitational force. If the speed is much higher, $\beta = v/c >> 10^{-3}$, the neutrino can conserve the helicity, keeping its direction of the momentum.

When a relic neutrino or antineutrino is reflected at a boundary surface of a material, its helicity can be reversed and then the left-handed neutrino becomes right-handed and the right-handed antineutrino becomes left-handed. In such situations they conserve spin, but not helicity. It is because the spin matrix $\vec{\sigma}$ commutes with the nonrelativistic Hamiltonian. Since the neutrino spin does not change by reflection, the neutrino helicity is reversed with a probability $\sin^2(\phi/2) = \sin^2\theta$ where $\phi = 2\theta$ is the angle between the incident and outgoing momenta of the neutrino (see Fig.1 for the grazing angle θ and see Appendix C for the helicity flip probability). If a relic neutrino is massive, it interacts with matter with the same strength irrespective of its helicity and a right-handed relic neutrino can be totally reflected at a boundary surface of matter just in the same way as a left-handed relic neutrino. Therefore if we trap massive relic neutrinos with kinetic energy $\varepsilon < U$ in a closed space surrounded by materials, they can be trapped for long time in spite of their frequent helicity flip due to the total reflection at the boundary surfaces of the space and an equilibrium state between left- and right-handed relic neutrinos will be established inside the closed space. Similar phenomena occur for massive antineutrinos trapped in a closed space. When the repulsive potential is too high, so high as $U > 2m_\nu$, the total reflection does not necessarily occurs and a special care should be paid (see Appendix B). Especially if relic neutrinos are massless, neutrinos with energy $E < U$ are not necessarily totally reflected at boundry surfaces of materials (see Appendix B), and can pass through the materials. Thus these ultra low energy massless neutrinos cannot be trapped inside a closed space surrounded by materials.

4. Detection of relic neutrinos by use of total reflection of neutrinos

One standard way to observe relic neutrinos by laboratory experiments may be to observe the mechanical effect of relic neutrinos upon a macroscopic material through its recoil movement due to forces exerted upon the material by relic neutrinos. Although the forces had been calculated in many previous papers[4], we shall roughly estimate the magnitude of forces exerted by relic neutrinos upon materials in order to show how difficult it is to observe the effects of the forces.

Instead of directly calculating the forces due to reflection and refraction of relic neutrinos by a macroscopic material, we use the interaction energy of relic neutrinos with a material since its spatial derivatives give the force exerted by relic neutrinos upon the material[4]. If we denote the potential energy for neutrinos in the i-th mass eigenstate as U_i and the number densities of i-th neutrino and antineutrino as ρ_{ν_i} and $\rho_{\bar{\nu}_i}$, respectively, we have the interaction energy $\tilde{U}_{int}$ of relic neutrinos and the material

$$\tilde{U}_{int} = \sum_i \int d^3x U_i(x)\left(\rho_{\nu_i}(x) - \rho_{\bar{\nu}_i}(x)\right), \tag{54}$$

where the integration is over the whole volume inside the material.

Relic neutrino and antineutrino densities are supposed to be uniform in space unless they are modified by presence of nearby materials, which deflect and diffract neutrinos and antineutrinos. If the number densities vary in space, the interaction energy $\tilde{U}_{int}$ depends on where the material is placed, and a small displacement of the material changes the value of $\tilde{U}_{int}$. The force $\vec{F}$ exerted by relic neutrinos and antineutrinos upon the material is given as the negative of the spatial derivative of $\tilde{U}_{int}$

$$\vec{F} = -\sum_i \int d^3x U_i(x)\left(\vec{\nabla}\rho_{\nu_i}(x) - \vec{\nabla}\rho_{\bar{\nu}_i}(x)\right). \tag{55}$$

It is clear that there should be no force exerted by relic neutrinos, if the neutrino and antineutrino densities are uniform in space or the neutrino and antineutrino densities are equal for each mass eigenstate of neutrino.

If we make the partial integration, $\vec{F}$ can be expressed as

$$\vec{F} = \sum_i \int d^3x \vec{\nabla} U_i(x)\left(\rho_{\nu_i}(x) - \rho_{\bar{\nu}_i}(x)\right) \tag{55'}$$

Since $-\vec{\nabla} U_i(x)$ is the force applied upon an i-th neutrino at position x by the material, $\vec{\nabla} U_i(x)$ is the recoil force applied upon the material by the i-th neutrino. Because the sign of U_i is reversed for antineutrinos, $-\vec{\nabla} U_i(x)$ is the force exerted by an i-th antineutrino upon the material. Therefore $\vec{F}$ given by the equation (55') is the sum of forces exerted by relic neutrinos and antineutrinos upon the material. For a uniform material $\vec{\nabla} U_i(x)$ is non-zero only at boundary surfaces of the material where reflection and refraction of neutrinos and antineutrinos take place and the associated momentum transfers to the material occur.

Although there is no known practical way to produce density gradients of relic neutrinos and antineutrinos, suppose we have succeeded in making spatial gradients of relic neutrino and antineutrino densities by using some neutrino lens systems. If we have such density gradient that is along the z-direction and is proportional to the density itself with a proportional constant which is opposite for neutrinos and antineutrinos,

$$\partial\rho_{\nu_i} / \partial z = \alpha_i \rho_{\nu_i}, \quad \partial\rho_{\bar{\nu}_i} / \partial z = -\alpha_i \rho_{\bar{\nu}_i}, \qquad (i = 1,2,3) \tag{56}$$

then we obtain the force applied on the material,

$$F_z = -\sum_i \int d^3x \alpha_i U\left(\rho_{\nu_i} + \rho_{\bar{\nu}_i}\right), \tag{57}$$

where α_i is the slope parameter for the density gradient of i-th neutrino. In order to estimate the order of magnitude of the force, suppose that the neutrino density has a small density gradient for ν_1 and no density gradient for other neutrino species. Then the force is given by $F_z \approx -2\alpha_1 U_1 \rho_{\nu_1} V_m$, where V_m is the total volume of the material.

Since U_1 is proportional to the atomic density of the material, $U_1 V_m$ is proportional to the total mass M of the material and therefore acceleration of the material due to the force by relic neutrinos is independent of the mass of the material. With $\alpha_1 = 10^{-4}/cm$, $\rho_{\nu_1} = 56/cm^3$, and $U_1 = 1.2\times 10^{-14} eV$ (for iron) for example, the acceleration a_z is quite small and given by

$$a_z = -2\alpha_1 U_1 \rho_{\nu_1} V / M = -2\alpha_1 U_1 \rho_{\nu_1} / \rho_M \sim 2\times 10^{-29} cm/sec^2 \qquad (58)$$

where ρ_M is the mass density of the material ($= 7.87 \mathrm{gr/cm^3}$ for iron). Possibility to measure such a small acceleration is quite remote. Conventional Cavendish-type torsion balance can measure[12] acceleration down to $10^{-13} cm/sec^2$ and possible improvements with a currently available technology to a sensitivity of $\sim 5\times 10^{-24} cm/sec^2$ had been proposed[13]. However even with such an improved sensitivity it is difficult to measure the effects of relic neutrinos upon matter.

Trapping relic neutrinos in a space surrounded by materials through total reflection of neutrinos at boundary surfaces might be one of promising ways to make a density gradient of relic neutrinos. Since we need very thick materials for the total reflection to occur, it is somewhat interesting to study what happens to relic neutrinos inside a cavity deep underground such as the neutrino laboratory at Kamioka. The crest near the surface of the earth has an average density of about $2.7\,\mathrm{g/cm^3}$ and it mainly consists of SiO_2, Al_2O_3, Fe_2O_3, MgO and CaO. For simplicity let us assume that the rock consists of only SiO_2. A molecule SiO_2 has $N = Z = 30$ and mass $\sim 1.0\times 10^{-22} g$. The number density of SiO_2 molecules is estimated to be $\rho_{SiO_2} \sim 2.7\times 10^{22}/\mathrm{cm^3}$. If we assume that the three kinds of neutrinos are all massive, we obtain the potential for mass eigenstate ν_i in the crest near the surface with Eq.(40),

$$U = 2K(G_F / 2\sqrt{2})\rho_{SiO_2}(-30 + 60\cdot \mathrm{dia}(|W_{e1}|^2, |W_{e2}|^2, |W_{e3}|^2)) \qquad (59)$$

Here we have the diagonal matrix "dia" due to the argument given in Section 2-2-1. If we use the approximate MNS matrix elements given in Eq.(44),

$$W_{e1} = \sqrt{2/3}, \quad W_{e2} = \sqrt{1/3}, \quad W_{e3} = 0,$$

we get

$$U = 2K\times 10^{-14} eV \cdot \mathrm{dia}(0.83, -0.83, -2.5) \qquad (60)$$

Since U is positive for ν_1, $\bar{\nu}_2$ and $\bar{\nu}_3$, ultra low energy ν_1, $\bar{\nu}_2$ and $\bar{\nu}_3$ with $\varepsilon < U$ and $2m > U$ are totally reflected when they enter from the atmosphere into the mountain over Kamioka neutrino laboratory and cannot reach the laboratory, while other kinds of neutrinos and antineutrinos for which $U < 0$ enter freely into the laboratory and pass through it. Thus it may seem that ultra low energy ν_1, $\bar{\nu}_2$ and $\bar{\nu}_3$ are missing inside Kamioka cave. However, these ultra low energy neutrinos and antineutrinos can still come into the cave through tunnels leading to the cave, after frequently repeating total reflection whenever they collide with walls of the tunnels. Once they reach the cave they are trapped inside the cave for a certain length of time due to total reflection at walls of the cave until they escape through the tunnels to outside space. Let us consider the helicities of ultra low energy relic neutrinos ν_1, $\bar{\nu}_2$ and $\bar{\nu}_3$ trapped inside the cave. Since they collide repeatedly and rapidly with walls of the cave thereby changing their helicity with a finite probability, we expect that an equilibrium state between left-handed and right-handed neutrinos with $\rho^{Cav}{}_{\nu\text{-left}} = \rho^{Cav}{}_{\nu\text{-right}}$ for $\nu = \nu_1,\ \bar{\nu}_2$, and $\bar{\nu}_3$ will be realized inside the cave. Here ρ^{Cav} is the number densitiy of ultra low energy neutrinos in the mine cavity. On the other hand $\rho_\nu{}^{Atm}$, which stands for the number density of ultra low energy neutrinos in outer space (atmosphere), we might have $\rho^{Atm}{}_{\nu_1-\text{left}} > \rho^{Cav}{}_{\nu_1-\text{left}}$ and $\rho^{Cav}{}_{\nu_1-\text{right}} > \rho^{Atm}{}_{\nu_1-\text{right}}$, if the equilibrium of the both helicities is not complete for neutrinos in outer space. Thus density gradients for $\rho_{\nu_1-\text{left}}$ and $\rho_{\nu_1-\text{right}}$ might be formed along the tunnels with left-handed ν_1 flowing into the cave and right-handed ν_1 flowing out from the cave through tunnels by neutrino diffusion processes. Similar density gradients might be formed for ultra low energy $\bar{\nu}_2$ and $\bar{\nu}_3$. In equilibrium these flows of neutrino in opposite directions will be equal in magnitude. Then the total density $\rho^{Cav}{}_{\nu_1-\text{left}} + \rho^{Cav}{}_{\nu_1-\text{right}}$ is the same everywhere along the tunnels and no density gradient will be formed for the total density of ν_1, and that is also true for $\bar{\nu}_2$ and $\bar{\nu}_3$. Thus no force will be applied by neutrinos upon a macroscopic material placed in the tunnels. It should be noted that the equilibrium of left-handed and right-handed neutrinos for very low velocity neutrinos ($\beta < 10^{-3}$) might have been made even before the neutrinos come into the tunnels of the cave, because of the gravitational bending of the neutrino trajectory in the space as discussed in the Introduction of this paper.

5. Concluding remarks

Neutrino interaction with matter is very weak and neutrinos pass through matter almost freely without being disturbed. Yet neutrinos experience a small potential U of the order of $10^{-14}\text{eV} \sim 10^{-13}\text{eV}$ in a medium. The sign of U depends on the kind of neutrinos, and ultra low energy massive relic neutrinos with kinetic energy $\varepsilon < U$ are totally reflected when they collide with the medium for which $0 < U < 2m_\nu$, while masssless relic neutrinos with energy E less than U can penetrate through the medium even without being reflected in the vertical collision with a wall. For the total reflection of ultra low energy massive neutrinos to occur the thickness of medium must be at least more than few meters long. Trapping ultra low energy relic neutrinos in a cavity surrounded by thick materials such as a deep underground laboratory was discussed thereby looking for a possible formation of density gradients of relic neutrinos with the negative result. Although detectability of relic neutrinos in laboratory experiments is quite remote at present and finding a practical way to make density gradients of relic neutrinos is quite difficult, we still hope that the total reflection of relic neutrinos by materials turns out to become useful in producing density gradient of relic neutrinos and density asymmetry of neutrino and antineutrino, and in detecting relic neutrinos in a remote future.

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Appendix .A Total reflection of relic neutrinos by a strong repulsive potential

The weak potential U for relic neutrinos in matter is expected to be of the order of $10^{-13} \sim 10^{-14} eV$ (see Eq.(39')) and either positive or negative depending on the type of neutrinos. It has a matrix form $U = V(1-\gamma^5)$ and effectively $U = V$ for nonrelativistic massive neutrinos and $U = 2V$ for massless neutrinos. If the potential is positive (repulsive) and larger than twice the mass m of the neutrino,

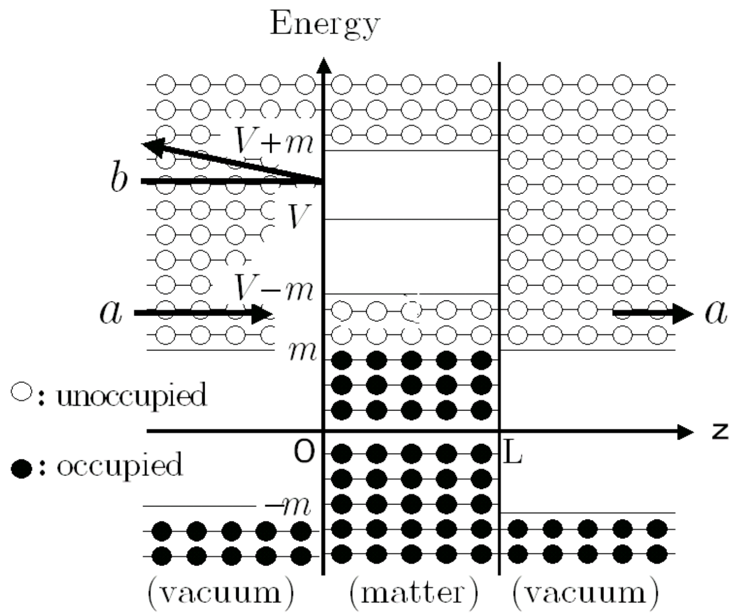
$$V > 2m,$$

a peculiar situation happens for neutrinos: they are not totally reflected at the boundary surfaces of the material and can pass through the material, even when the incident energy is less than the potential height. This happens also for massless neutrinos in a repulsive potential. This phenomenon is explained in the following.

Now suppose that a neutrino with energy E satisfying $m < E < V - m$ impinges upon a material where the neutrino feels a repulsive potential V. Although the potential is highly repulsive, the neutrino in the above energy range can still enter into the material by transiently occupying such a level of neutrino in the material that has the same energy as the incident one and is unoccupied by neutrinos. Thus the repulsive potential acts as a filter selectively allowing neutrinos in the above energy range to pass through the material without being totally reflected at the boundary surface of the material. This phenomenon is illustrated in Fig.A1.

Fig.A1. Energy levels in the matter with strong repulsive potential:

The energy levels in the matter for $U = V(1-\gamma^5)$ consist of two regions, $E \geq V + m$ and $E \leq V - m$ In case of strong potential (or massless case) satisfying $V > 2m$, (a) the neutrino of with $m < E_a < V - m$ can penetrate the matter, while (b) the neutrino with $V - m < E_b < V + m$ cannot. This is due to whether there is a corresponding energy level in the matter or not. The energy levels with $E < V - m$ in the matter are analogous to the negative energy levels in the vacuum. The repulsive potential acts as a filter if it is thick enough $L >> 1/|p'_z|$.



In the figure A1 an incident neutrino with energy E_a is allowed to enter the matter, while the one with E_b is not. This is because there is an energy level in the matter with energy eigenvalue equal to E_a, while there is no energy level in the matter with energy equal to E_b. The former is analogous to the negative energy level in the vacuum, though the energy is positive in the matter. The masses of the heaviest two neutrinos are estimated to be of the order of $10^{-1} \sim 10^{-2} eV$, which are much larger than the weak potential V. However, if one of the three kinds of neutrinos is massless, any small positive potential V satisfies the condition $V > 2m$. Therefore the situation like Fig.A1 applies for a massless neutrino and the massless relic neutrinos with energy E less than V can pass through a material without being totally reflected at the boundary surface of the material. In fact it turns out that these massless neutrinos entering vertically onto the matter surface can pass through the material without being reflected at all. In order to show the above mentioned phenomena to occur we shall calculate the rates of reflection and transmission for the incoming neutrino with energy E ($m < E < V - m$) and momentum p in case of uniform repulsive potential V larger than $2m$ with finite thickness L. We assume for simplicity that the incoming neutrino momentum is along z-direction which is perpendicular to the material surfaces. Then the process is reduced to one-dimensional problem. We start with the Dirac equation for effective Hamiltonian H given by Eq.(27), thereby taking

$\vec{p} = (0, 0, p)$ for the incoming neutrino wave (A1)

$\vec{p} = (0, 0, -p)$ for the reflected neutrino wave (A2)

$\vec{p}'_a = (0, 0, p'_a)$ for one of the neutrino waves in the matter (A3a)

$\vec{p}'_b = (0, 0, p'_b)$ for the other of the neutrino waves in the matter (A3b)

$\vec{p} = (0, 0, p)$ for the transmitted neutrino in the vacuum (A4)

The potential $U(z) = V(z)(1 - \gamma^5)$ is given by

$V(z) = 0$ for $z < 0$, and $z > L$ $V(z) = V = Const. > 0$ for $0 \le z \le L$.

We denote the normalized wave functions of the incoming neutrino, the reflected neutrino, and the transmitted neutrino in the vacuum as ψ_1, ψ_2 and ψ_3, respectively. The normalized wave functions in the matter are denoted as ψ_a and ψ_b, though it is not specified which of them is going forward or backward. In the present situation the neutrino spin in z-direction should be conserved, since the total angular momentum in the z-direction is conserved and there is no contribution of orbital angular momentum in z-direction. We therefore use the same spin spinor χ for all five wave functions. We assume $\sigma_z \chi = h\chi$ ($h = \pm 1$). The Hamiltonian given in Eq.(27) is simplified by using the helicity eigenstate with $\vec{\sigma}\vec{p}=hp_z$, and is rewritten as

$$H=\begin{pmatrix} 0 & p_z h \\ p_z h & 0 \end{pmatrix}+m\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}+V(z)\left(1-\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right) \quad . \tag{A5}$$

This gives the eigenvalue equation,

$$\left(E-V(z)\right)\psi=\begin{pmatrix} 0 & h(p_z-hV(z)) \\ h(p_z-hV(z)) & 0 \end{pmatrix}\psi+m\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\psi \tag{A5'}$$

The Dirac equation in matter is obtained from the free equation with the substitution

$$E\to E-V\,,\quad p\to p'-hV\,. \tag{A6}$$

The energy-momentum relation in the matter is derived by the substitution rules (A6) from the free one $E^2=m^2+p^2$ as

$$(E-V)^2=m^2+(p'-hV)^2\,. \tag{A7}$$

The free wave functions ψ_1, ψ_2 or ψ_3 are given as

$$\psi_1=N\begin{pmatrix} \chi \\ \dfrac{hp}{E+m}\chi \end{pmatrix}e^{ipz-iEt}\,,\quad \psi_2=N\begin{pmatrix} \chi \\ -\dfrac{hp}{E+m}\chi \end{pmatrix}e^{-ipz-iEt}\,,\quad \psi_3=N\begin{pmatrix} \chi \\ \dfrac{hp}{E+m}\chi \end{pmatrix}e^{ipz-iEt} \tag{A8}$$

with

$$N=\sqrt{\frac{E+m}{2E}}\,.$$

The wave functions in the matter, ψ_a and ψ_b are given by the substitution rules (A6) from the free one as

$$\psi_a=N'\begin{pmatrix} \chi \\ \dfrac{hp'_a-V}{E+m-V}\chi \end{pmatrix}e^{ip'_a z-iEt}=N'\begin{pmatrix} \chi \\ \dfrac{\sqrt{(E-V)^2-m^2}}{E+m-V}\chi \end{pmatrix}e^{ip_a{}' z-iEt} \tag{A8'}$$

$$\psi_b=N'\begin{pmatrix} \chi \\ \dfrac{hp'_b-V}{E+m-V}\chi \end{pmatrix}e^{ip'_b z-iEt}=N'\begin{pmatrix} \chi \\ -\dfrac{\sqrt{(E-V)^2-m^2}}{E+m-V}\chi \end{pmatrix}e^{ip'_b z-iEt} \tag{A8''}$$

with

$$N'=\sqrt{\frac{E-V+m}{2(E-V)}} \tag{A9}$$

In deriving Eqs.(A8') and (A8") we have used the relation Eq.(A7) for p'_a and p'_b

$$\sqrt{(E-V)^2-m^2}=p'_a h-V \tag{A10}$$

and

$$\sqrt{(E-V)^2-m^2}=V-p_b'h\,. \tag{A10'}$$

Here in Appendix A we have taken the space volume equal to unity for simplicity. We put the wave function $\psi(z,t)$ as

$$\psi=\psi_1+R\psi_2 \qquad \text{for} \qquad z<0 \tag{A11}$$

$$\psi=A\psi_a+B\psi_b \qquad \text{for} \qquad 0\le z\le L \tag{A11'}$$

and

$$\psi=T\psi_3 \qquad \text{for} \qquad z>L\,, \tag{A11''}$$

where R, A, B and T are some constants. The continuity of $\psi(z,t)$ at $z=0$, and $z=L$ gives the constraint

$$\psi_1+R\psi_2=A\psi_a+B\psi_b \qquad \text{at} \qquad z=0 \tag{A12}$$

$$A\psi_a+B\psi_b=T\psi_3 \qquad \text{at} \qquad z=L \tag{A12'}$$

The conditions (A12) and (A12') determines R and T, when we consider the continuity of the upper and the lower components of Dirac spinors. We have

$$1+R=a+b \tag{A13.1}$$

$$S(1-R)=a-b \tag{A13.2}$$

$$T=\tilde{a}+\tilde{b} \tag{A13.3}$$

$$ST=\tilde{a}-\tilde{b} \tag{A13.4}$$

with

$$a=(N'/N)A,\ b=(N'/N)B,\ \tilde{a}=e^{ip_a'L}a,\ \tilde{b}=e^{ip_b'L}b\,, \tag{A14}$$

and

$$S=\frac{ph(E+m-V)}{(E+m)\sqrt{(E-V)^2-m^2}} \tag{A14'}$$

We have from Eqs.(A13.1), (A13.2), (A13.3) and (A13.4)

$$R=\frac{(1-S^2)(e^{i(p_a'-p_b')L}-1)}{(1+S)^2-(1-S)^2e^{i(p_a'-p_b')L}}\,. \tag{A15}$$

and

$$T=e^{ip_a'L}\frac{4S}{(1+S)^2-(1-S)^2e^{i(p_a'-p_b')L}}\,. \tag{A16}$$

We obtain the reflection rate r and the transmission rate t as

$$r=|R|^2=\frac{(1-S^2)^2(1-\cos\kappa L)}{(1+6S^2+S^4)-(1-S^2)^2\cos\kappa L} \tag{A17}$$

$$t=|T|^2=\frac{8S^2}{1+6S^2+S^4-(1-S^2)^2\cos\kappa L} \tag{A18}$$

with S given in Eq.(A14') and κ defined by

$$\kappa = h(p'_a - p'_b) = 2\sqrt{(E-V)^2 - m^2} \tag{A19}$$

We have the conservation of the neutrino number

$$r + t = 1 \tag{A20}$$

as expected. We see that the transmission rate does not vanish for a strong repulsive potential that satisfies $V > E + m$, which gives real $(p' - hV)$ from (A7), and we see further that the reflection rate vanishes for the vertically impinging massless neutrino,

$$r = 0 \, , \tag{A21}$$

since Eq.(A14') gives $S^2 = 1$ for $m = 0$, which leads to $R = 0$ in Eq.(A15). This vanishing of the reflection rate is also understood by considering the conservation of chirality and angular momentum, as is argued in Section 3.

We also note that such "negative energy level" (though it has actually positive energy due to large positive potential, we call here it so merely as an analogy to the negative energy level in the vacuum) has a peculiar property of velocity. Let us consider the case of massless neutrino with repulsive potential for simplicity. The energy eigenvalue is given for such a "negative energy state" by

$$E(p') = U - p' \tag{A22}$$

with $U = 2V$ (see Eq.(A7) with $h = -1$). The dispersion relation of (A22) gives the group velocity as

$$\vec{v} \equiv \frac{\partial E}{\partial \vec{p}'} = -\frac{\vec{p}'}{p'} \quad , \tag{A23}$$

which shows that the direction of the group velocity is opposite to the momentum direction in matter for " negative energy state".

Appendix B. Potential for Majorana relic neutrinos in matter

We consider the potential for relic neutrinos in matter when the neutrino is a massive Majorana neutrino instead of a Dirac neutrino. The Majorana neutrino is equivalent to a two-component Weyl neutrino and there is no distinction between the neutrino and the antineutrino. Although a Majorana field has two components instead of four components, we shall here express the Majorana field as a four-componet field just like a four-component Dirac field.

B1. Definitions and the Lagrangian

The free Dirac equation for Majorana neutrino field is written as

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \qquad \text{(B1)}$$

In case of Majorana field it is convenient to use the following representation for Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \qquad \text{(B2)}$$

instead of the conventional one defined to derive Eq.(28), for which γ^0 is defined by γ^5 of Eq.(B2). Denoting γ matrices defined to derive Eq.(28) as γ_D, we can express the γ matrices of Eq.(B2) by the unitary transformation, $\gamma^\mu = A\gamma_D{}^\mu A^{-1}$ with $A = (1+\gamma_D{}^5\gamma_D{}^0)/\sqrt{2} = (1+\gamma^5\gamma^0)/\sqrt{2}$.

We impose the condition of Majorana field under the charge conjugation,

$$\psi = \psi^c \qquad \text{(B3)}$$

where the charge conjugation of ψ is defined as (writing the transposition by the symbol tr)

$$\psi^c = C\bar{\psi}^{tr} \quad \text{with} \quad C = i\gamma^0\gamma^5 \qquad \text{(B4)}$$

Writing ψ with two component spinors ϕ and η as

$$\psi = \begin{pmatrix} \phi \\ \eta \end{pmatrix} \qquad \text{(B5)}$$

we have from Eq.(B3)

$$\eta = i\sigma^2\phi^{\dagger tr} \equiv i\sigma^2\phi^*, \quad \eta^\dagger = -i\phi^{tr}\sigma^2 \qquad \text{(B6)}$$

which shows η is not independent of $\phi^\dagger$. The Lagrangian which gives Eq.(B1) is

$$L = \frac{1}{2}(i\gamma^\mu\partial_\mu - m)\psi, \qquad \text{(B7)}$$

with a factor 1/2 reflecting the linear dependence of $\bar{\psi}$ on ψ through Eq.(B3). Substituting Eq.(B5) with Eq.(B6) into Eq.(B7). we have the expression for the free Lagrangian in terms of the 2 dimensional spinor ϕ as

$$L = \phi^\dagger(i\partial_0 - i\vec{\sigma}\vec{\partial})\phi - \frac{i}{2}m(\phi^{tr}\sigma^2\phi - \phi^\dagger\sigma^2\phi^*) \qquad \text{(B7')}$$

B2. Weak current

Let us examine the property of weak current

$$j^\mu = \bar{\psi}\gamma^\mu(1-\gamma^5)\psi \qquad \text{(B8)}$$

in case of Majorana field. Using the condition Eq.(B3) we can rewrite the Eq.(B7) as

$$j^\mu = \bar{\psi}\gamma^\mu(-1-\gamma^5)\psi. \qquad \text{(B8')}$$

Comparing Eq.(B8) with Eq.(B8'), we see that the vector part of the weak current vanishes and only the axial vector part remains.

B3. Matrix elements

Next let us obtain the expectation values or matrix elements for the Majorana field. Let us denote the creation and annihilation operators of neutrino, as $a^{\dagger}{}_{k,h}$ and $a_{k,h}$, which satisfy the conventional anti-commutation relations,

$$\{a_{k,h}, a^{\dagger}{}_{k',h'}\} = \delta_{kk'}\delta_{hh'}, \quad \{a_{k,h}, a_{k',h'}\} = 0, \quad \{a^{\dagger}{}_{k,h}, a^{\dagger}{}_{k',h'}\} = 0. \tag{B9}$$

Here k and h are the four-momentum and the helicity of neutrino, respectively. We can expand the Majorana field with $a^{\dagger}{}_{k,h}$ and $a_{k,h}$ as

$$\psi = \sum_{k,h}(a_{k,h}e^{ikx}u + a^{\dagger}{}_{k,h}e^{-ikx}v), \tag{B10}$$

The wave functions $e^{ikx}u$ and $e^{-ikx}v$ satisfy the Dirac equation (B1) which gives

$$(\not{k} - m)u = 0 \tag{B11}$$

$$(\not{k} + m)v = 0 \tag{B12}$$

and also the Majorana field condition Eq.(B3) which gives

$$u = C\overline{v}^{tr} \tag{B13}$$

$$v = C\overline{u}^{tr} \tag{B14}$$

The normalization consistent with Eq.(B9) should be given by

$$\int d^3xu^{\dagger}u = \int d^3xv^{\dagger}v = 1 \tag{B15}$$

This can be rewritten as

$$\begin{aligned} u &= \sqrt{\frac{1}{V}}\hat{u} \qquad \text{with} \qquad \hat{\overline{u}}\hat{u} = 1 \\ v &= \sqrt{\frac{1}{V}}\hat{v} \qquad \text{with} \qquad \hat{\overline{v}}\hat{v} = 1 \end{aligned}, \tag{B16}$$

with the space volume V.

The equations (B9), (B10), (B13) and B(14) give the matrix elements

$$\begin{aligned} &< 0|\psi(x)|k,h> = < 0|\psi(x)a^{\dagger}{}_{kh}|0> = e^{ikx}u \\ &< 0|\overline{\psi}(x)|k,h> = e^{ikx}\overline{v} \\ &< k,h|\overline{\psi}(x)|0> = e^{-ikx}\overline{u} \\ &< k,h|\psi(x)|0> = < 0|a_{kh}\psi(x)|0> = e^{-ikx}v \end{aligned} \tag{B17}$$

These matrix elements of the fields give the matrix element of the operator

$$G(x) = \overline{\psi}G\psi \tag{B18}$$

with an arbitrary 4×4 matrix G as

$$\left\langle k',h'\middle|G(x)\middle|k,h\right\rangle = \overline{u}(k',h')Gu(k,h)e^{ikx-ik'x} - \overline{v}(k,h)Gv(k',h')e^{ikx-ik'x} \quad \text{(B19)}$$

where the negative sign of the second term of the right-hand side is due to the anti-commutation relation of the fermi operators. The equation (B19) is rewritten with use of Eqs.(B13) and (B14) as

$$\left\langle k',h'\middle|G(x)\middle|k,h\right\rangle = \overline{u}(k',h')Gu(k,h)e^{ikx-ik'x} + \overline{u}(k',h')C^{\dagger}G^{tr}Cu(k,h)e^{ikx-ik'x} \quad \text{(B20)}$$

$$= \overline{u}(k',h')[G + C^{\dagger}G^{tr}C]u(k,h) \quad \text{(B21)}$$

Noting $C^{\dagger}(\gamma^{\mu})^{tr}C = -\gamma^{\mu}$ and $C^{\dagger}(\gamma^{\mu}\gamma^{5})^{tr}C = \gamma^{\mu}\gamma^{5}$, we have the matrix elements of the weak current for the Majorana particle as

$$\left\langle k,h\middle|\overline{\psi}\gamma^{\mu}\psi\middle|k,h\right\rangle = 0 \quad \text{(B22)}$$

$$\left\langle k,h\middle|\overline{\psi}\gamma^{\mu}\gamma^{5}\psi\middle|k,h\right\rangle = 2\overline{u}\gamma^{\mu}\gamma^{5}u\,. \quad \text{(B23)}$$

We thus obtain the expectation value of the weak potential

$$<\int d^3x\, V\, \overline{\psi}\gamma^{\mu}(1-\gamma^{5})\psi> = (-2)V\int d^3x\overline{u}\gamma^{\mu}\gamma^{5}u = (-2)V\hat{\overline{u}}\gamma^{\mu}\gamma^{5}\hat{u} \quad \text{(B24)}$$

Denoting the 3-momnetum in matter as $\vec{p}'$, we obtain the Dirac equation for Majorana field in matter,

$$E\hat{u} = (\vec{\alpha}\,\vec{p}' + m\gamma^{0} - 2V\gamma^{5})\,\hat{u}\,. \quad \text{(B25)}$$

Here we note that the vector part of the interaction term vanishes for Majorana field, and the axial vector part coming from the interaction energy density $V\overline{\psi}\gamma^{0}(-\gamma^{5})\psi$ gives a factor 2 larger interaction term compared with the Dirac field case, as is also evident in Eq.(B23). As was explained in Section 2.2.1, the equation (B25) is just the free equation with the displacement

$$p' \to p' - 2h'V$$

with the heliccity in matter h' ($h' = \pm 1$). This replacement easily derives the energy momentum relation in the matter (see also Eq.(30M)) from the free one as,

$$E^2 = m^2 + (p' - 2h'V)^2\,, \quad \text{(B26)}$$

and the wave function $\hat{u}$ in the matter from the free one as

$$\hat{u} = N'\begin{pmatrix} \chi \\ \dfrac{(p'-2h'V)h'}{E+m}\chi \end{pmatrix}, \quad \text{with} \quad N' = \sqrt{\frac{E+m}{2E}} \quad \text{(B27)}$$

The equations (B26) and (B27) give

$$\hat{\overline{u}}\gamma^{0}\gamma^{5}\hat{u} = \frac{(p'-2h'V)h'}{E} \quad \text{(B28)}$$

and

$$\hat{\bar{u}}\vec{\gamma}\gamma^5\hat{u} = \frac{(p' - 2h'V)}{E} h'\chi^\dagger\vec{\sigma}\chi \qquad \text{(B29)}$$

Using the relations $\vec{\sigma}\vec{p}'\chi = p'h'\chi$ and $\chi^\dagger\vec{\sigma}\vec{p}' = \chi^\dagger p'h'$, we get

$$\chi^\dagger\sigma^j\chi = h'p'^j / p' \qquad \text{(B30)}$$

This substituted in Eq.(B29) gives

$$\hat{\bar{u}}\vec{\gamma}\gamma^5\hat{u} = \frac{(p' - 2h'V)}{E}\frac{\vec{p}'}{p'} \qquad \text{(B31)}$$

If we differentiate the both hand sides of Eq.(B26) with respect to $\vec{p}'$ and use the definition of the group velocity, $v^j = \partial E / \partial p^j$, we get

$$v'^j = \frac{\partial E}{\partial p'^j} = \frac{(p' - 2h'V)}{E} \cdot \frac{p'^j}{p'} \qquad \text{(B32)}$$

This equation gives

$$v' = | p' - 2h'V | / E \qquad \text{(B33)}$$

The equations (B28),(B31),(B32),(B33) give

$$\hat{\bar{u}}\gamma^0\gamma^5\hat{u} = \pm v'h' \qquad \text{(B34)}$$

$$\hat{\bar{u}}\vec{\gamma}\gamma^5\hat{u} = \vec{v}'. \qquad \text{(B35)}$$

In the equation (B34) the upper sign corresponds to $p' - 2h'V > 0$, and the lower sign to $p' - 2h'V < 0$; the latter is possible only with very large potential $|2V| > p$ .In conclusion the expectation value of the weak Hamiltonian is given by

$$< \int d^3x\bar{\psi}\gamma^\mu(1-\gamma^5)\psi > = < \int d^3x\bar{\psi}\gamma^\mu(-\gamma^5)\psi > = (-2)\hat{\bar{u}}\gamma^\mu\gamma^5\hat{u}$$

$$= (-2)(\pm v'h') \quad \text{for} \quad \mu = 0 \qquad \text{(B36)}$$

with upper sign for $p' > 2h'V$ and lower sign for $p' < 2h'V$

$$\text{or} \quad = (-2) \cdot v'^j \qquad \text{for} \quad \mu = j \ (=1,2,3) \qquad \text{(B37)}$$

We note that the expressions Eq.(B36) and Eq.(B37) depend only on the velcoity and the helicty and therefore are applicable to the free case by using the free velocity v and the free helicity h.

Appendix C Helicity flip probability

When a nonrelativistic neutrino coming from the vacuum hits the surface of a material with the grazing angle θ (see Fig.1), and is reflected at the surface, the helicity may flip, because the momentum is changed by the reflection but the spin is approximately conserved in the nonrelativistic reflection process. In order to estimate the probability of helicity flip, let us take the momentum of the incident neutrino in z-direction, $\vec{p} = (0, 0, p)$ and the momentum of the reflected neutrino in z-x plane,

$\vec{p}' = (p\sin\phi, 0, p\cos\phi)$. We have $\phi = 2\theta$ owing to the conservation of momentum parallel to the surface of the boundary. Since the spin is conserved, we can use the same spinor wave function χ_h for the incident neutrino and for the reflected one. Suppose the incident neutrino has the helicty h ($h = \pm 1$) satisfying,

$$\sigma_z \chi_h = h\chi_h \qquad \text{(C1)}$$

We have

$$\chi_{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \chi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{(C2)}$$

The helicity eigenstate of the reflected neutrino $\chi'_{h'}$ with helicity h' satisfies the equation

$$(\sigma_z \cos\phi + \sigma_x \sin\phi)\chi'_{h'} = h'\chi'_{h'} \qquad \text{(C3)}$$

It is straightforward to solve this equation and the solutions are

$$\chi'_{+1} = \begin{pmatrix} \cos(\phi/2) \\ \sin(\phi/2) \end{pmatrix}, \quad \chi'_{-1} = \begin{pmatrix} -\sin(\phi/2) \\ \cos(\phi/2) \end{pmatrix} \qquad \text{(C4)}$$

Thus the spinor of the reflected neutrino, which is the same as the spinor of the incident neutrino χ_h, satisfies the equation

$$\begin{aligned} \chi_{+1} &= \cos(\phi/2)\chi_{+1}' - \sin(\phi/2)\chi_{-1}' \\ \chi_{-1} &= \sin(\phi/2)\chi_{+1}' + \cos(\phi/2)\chi_{-1}' \end{aligned} \qquad \text{(C5)}$$

The equations (C5) show that the helicity conserving amplitude is $\cos(\phi/2)$ and the helicity flip amplitude is $\mp\sin(\phi/2)$. When two spinors ξ_1 and ξ_2 satisfy $\vec{\sigma}\hat{n}_i\xi_i = \lambda_i\xi_i$ ($\lambda_i = \pm 1$) with unit vectors $\hat{n}_i$ ($i = 1,2$), the above argument gives the product of them, apart from a phase factor, as

$$\xi_1^\dagger(\lambda_1)\xi_2(\lambda_2) = \sqrt{(1 + \lambda_1\lambda_2\hat{n}_1\hat{n}_2)/2} \qquad \text{(C6)}$$

The probability of helicity flip is obtained as $\sin^2(\phi/2)$. It can be written with the grazing angle θ as $\sin^2\theta$ owing to the relation $\phi = 2\theta$. Thus we have

$$\text{(the helicity flip probability)} = \sin^2(\phi/2) = \sin^2\theta\,. \qquad \text{(C7)}$$

Appendix D. Probability of mass changing reflection

D1. Dirac equation and wave functions

The off-diagonal elements of the weak potential Eq.(39) causes inelastic reflection at material surface, changing the mass eigenstate ν_i ($i = 1,2,3$). In order to estimate the rate of mass changing reflection,

$$\nu_i \to \nu_j \quad \text{with} \quad i \neq j,$$

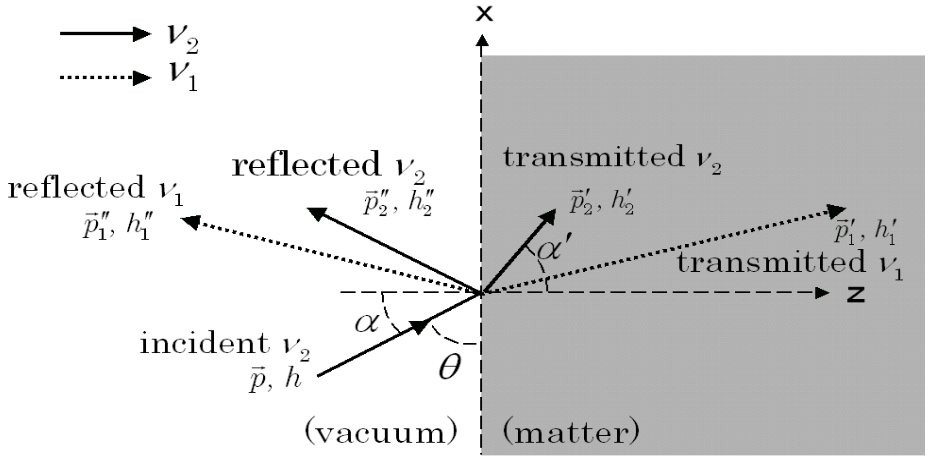
let us first take a simple model in which there are only two kinds of mass eigenstates ν_1 and ν_2, with masses m_1 and m_2, respectively (assuming $m_1 < m_2$), and consider in particular the exoergic reflection process,

$$\nu_2 \to \nu_1 .$$

Using the space coordinates x, y and z, we assume to have the vacuum for $z < 0$ and a homogenous matter for $z > 0$ with the boundary at $z = 0$ (see Fig.D1).

Fig.D1. The elastic or inelastic reflection and transmission

The incident neutrino ν_2 is reflected or transmitted at the boundary ($z = 0$) between the vacuum and the matter, and may or may not change to ν_1 at $z = 0$ due to off-diagonal weak potential.



We take the Hamiltonian,

$$H = I_m \otimes \vec{\alpha}\vec{P} + M \otimes \beta + \theta(z) V \otimes (1 - \gamma^5), \tag{D1}$$

where $\vec{P} = -i\vec{\nabla}$ is the momentum operator, $\theta(z)$ is the theta function defined by $\theta(z) = 0$ for $z < 0$ and $\theta(z) = 1$ for $z > 0$. In the equation (D1) we have direct product matrices of the form $A \otimes B$ with A having 2×2 components, operating on the space of neutrino species and B having 4×4 components, operating on the Dirac spinor. We use the definitions of Dirac matrices given in Eq.(28),

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \ \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{D2}$$

In the equation (D1) I_m is the 2×2 identity matrix, M is the diagonal mass matrix, and V is the potential matrix with both diagonal and off-diagonal elements,

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \quad , \quad V = \begin{pmatrix} a_1 & b \\ b & a_2 \end{pmatrix} \tag{D3}$$

We decompose the total Hamiltonian into diagonal and off-diagonal parts,

$$H = H_{dia} + H_{off} \tag{D4}$$

with

$$\begin{aligned} H_{dia} &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \vec{\alpha}\vec{p} + \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \otimes \beta + \theta(z) \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \otimes (1-\gamma^5) \\ H_{off} &= \theta(z) \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \otimes (1-\gamma^5) \end{aligned} \tag{D5}$$

We omit the symbol '$\otimes$' when the meaning is evident. We express the wave function Ψ as a linear combination of direct products of ζ (a 2-component spinor of neutrino species) and ψ (a 4-component Dirac spinor),

$$\Psi_{A\otimes\alpha} = \zeta_A \otimes \psi_\alpha . \quad (A = 1,2 \text{ and } \alpha = 1,2,3,4) \tag{D6}$$

The spinor ζ has the upper component for the mass eigenstate ν_1 and the lower one for ν_2. We sometimes express Ψ as a product of the space-time factor like $e^{i\vec{p}\vec{x}-iEt} / \sqrt{V_0}$ and the space-time independent factor Φ as

$$\Psi = \Phi e^{i\vec{p}\vec{x}-iEt} / \sqrt{V_0} , \tag{D7}$$

with the total space volume V_0.

D2. Solutions of the eigenvalue equation of the total Hamiltonian and the wave functions

D2-1. Solutions and the wave functions in the vacuum

In the vacuum ($z < 0$) the Hamiltonian (D1) is free and trivial. However, in order to clarify notations and definitions let us explicitly write the wave function of the neutrino with mass m_i, momentum $\vec{p}_i$, energy $E_i = \sqrt{p_i^2 + m_i^2}$ and helicity h_i as

$$\Psi_i(\kappa_i, \vec{x}, t) = \Phi_i(\kappa_i) e^{i\vec{p}_i\vec{x}-iE_it} / \sqrt{V_0} , \tag{D8}$$

with

$$\Phi(\kappa_i) = \zeta_i \otimes \phi_i(\kappa_i) . \tag{D9}$$

Here we denote a set of the eigenvalues as $\kappa_i = (E_i, p_i, h_i)$. The spinor ζ_i is defined to have the k-th component $(\zeta_i)_k = \delta_{ki}$ and represents ν_i. The wave function $\phi_i(\kappa_i)$ is the solution of the free Dirac equation for ν_i with the normalization $|\phi_i(\kappa_i)|^2 = \phi_i^\dagger \phi_i = 1$, and given by

$$\phi_i = N_i \begin{pmatrix} \chi_i \\ \frac{p_i h_i}{E_i + m_i} \chi_i \end{pmatrix}, \qquad N_i = \sqrt{(E_i + m_i)/2E_i} \tag{D10}$$

where χ_i is a 2-component spin spinor with normalization $\chi_i^\dagger \chi_i = 1$. We express the neutrino wave function in the vacuum as a linear combination of three types of waves, the wave of the incident ν_2 with momentum $\vec{p} = (p_x, 0, p_z)$, that of the reflected ν_2 with $\vec{p}_2'' = (p_x, 0, -p_z)$ and that of the inelastically reflected ν_1 with $\vec{p}_1'' = (p_x, 0, -p_z'')$, denoted as $\Psi_2(E, \vec{p})$, $\Psi_2(E, \vec{p}_2'')$ and $\Psi_1(E, \vec{p}_1'')$, respectively. All the three waves have the same energy

$$E = \sqrt{m_2^2 + p^2} = \sqrt{m_2^2 + p_2''^2} = \sqrt{m_1^2 + p_1''^2} \tag{D11}$$

and the same transverse momentum p_x, since p_x is conserved , as is explained in Section 3. We thus have the wave function for $z < 0$ as

$$\Psi = \Psi_2(E, \vec{p}, h) + \sum_{h_2''=\pm 1} R_2(h_2'')\Psi_2(E, \vec{p}_2'', h_2'') + \sum_{h_1''=\pm 1} R_1(h_1'')\Psi_1(E, \vec{p}_1'', h_1'') \tag{D12}$$

Here the coefficients $R_2(h_2'')$ and $R_1(h_1'')$ are constant.

D2-2 Solutions and the wave functions in the matter

In order to obtain the eigenfunctions of the total Hamiltonian H of Eq.(D1), we first solve, in section D2-2a, the eigenvalue equation for H_{dia} given by Eq.(D5), and next, in section D2-2b, we obtain eigenfunctions of H by perturbation with respect to the off-diagonal Hamiltonian H_{off} given by Eq.(D5), assuming $m_2 - m_1 >> |b|$. We determine a set of eigenfunctions up to the 1st order in b, and express the transmitted wave in the matter as a linear combination of these eigenfunctions.

D2-2a 0th order solutions in matter with b = 0

Each diagonal element of H_{dia} gives the Hamiltonian for the mass eigenstate ν_i,

$$H^{(i)} \equiv H_{dia}^{(i)} = \vec{\alpha}\vec{p}' + m_i\beta + \theta(z)a_i(1-\gamma^5), \qquad (i = 1,2) \tag{D13}$$

Taking account of the conservation of transverse momentum p_x, we take the momentum in the matter as $\vec{p}' = (p_x, 0, p_z')$. The energy eigenvalue E_i' and eigenfunctions $\phi_i^{(mat)}$ with helicity h_i' are obtained from the free solutions with the substitution rules $E_i \to E_i' - a_i$ and $p \to p_i' - h_i' a_i$ explained in (A6) of Appendix A, as

$$(E_i' - a_i)^2 = m_i^2 + (p_i' - h_i' a_i)^2 \tag{D14}$$

and

$$\phi_i^{(mat)}(\kappa_i) = N_i' \begin{pmatrix} \chi_i \\ \dfrac{(p_i' - h_i' a_i) h_i'}{E_i' - a_i + m_i} \end{pmatrix} , \qquad N_i' = \sqrt{(E_i' - a_i + m_i) \big/ 2(E_i - a_i)} . \qquad \text{(D15)}$$

Using Eqs.(D6) and (D7) we have the basis of eigenfunctions for $z > 0$ with b = 0

$$\Psi_i^{(mat)}(\kappa_i, x, t) = \Phi_i^{(mat)}(\kappa_i) e^{i\vec{p}_i{}'\vec{x} - iE_i t} / \sqrt{V_0} , \qquad \text{(D16)}$$

with $\Phi_i^{(mat)}(\kappa_i) = \zeta_i \otimes \phi_i^{(mat)}(\kappa_i)$. (D17)

D2-2b Solutions in matter with b ≠ 0 and the transmitted wave function

We use the perturbation method to obtain the eigenfunctions of the total Hamiltonian H up to the 1st order in H_{off}. We denote the eigenfunction of H with an eigenvalue set $\kappa_i' = (E_i', \vec{p}_i', h_i')$ as $\boldsymbol{\Psi}^{(matt)}(\kappa_i')$, and expand it using the basis set of the eigenfunctions $\Psi_i^{(mat)}$ given by Eq.(D16). We put

$$\boldsymbol{\Psi}_i^{(matt)} = \boldsymbol{\Phi}_i^{(mat)} e^{i\vec{p}_i{}'\vec{x} - iE_i t} / \sqrt{V_0} , \qquad \text{(D18)}$$

and in the 0th order of H_{off} we take

$$\boldsymbol{\Phi}_i^{(mat)}(\kappa_i') = \Phi_i^{(mat)}(\kappa_i') \qquad \text{(D19)}$$

The eigenvalue set κ_i' satisfies Eq.(D14)

$$(E_i' - a_i)^2 = m_i^2 + (p_i' - h_i' a_i)^2 , \qquad \text{(D20)}$$

since the energy eigenvalue has no correction in the 1st order of b. We take

$$E_i' = E . \qquad \text{(D21)}$$

where E is the energy of the incident neutrino given by $E = \sqrt{m_2^2 + p^2}$.

The energy eigenstate up to the 1st order in H_{off} is given by the standard perturbation method as a linear combination of eigenstates of H_{dia} ,

$$\boldsymbol{\Phi}_i^{(mat)}(\kappa_i') = \Phi_i^{(mat)}(\kappa_i') + \beta_{ij} \Phi_j^{(mat)}(\tilde{\kappa}_{ij}') \quad (i \neq j) \qquad \text{(D22)}$$

with $\tilde{\kappa}_{ij}' = (\tilde{E}_j', \tilde{\vec{p}}_j, \tilde{h}_j')$. (D23)

We note that the added wave function $\Phi_j^{(mat)}(\tilde{\kappa}_j')$ has the same momentum and spin as the 0th order solution $\Phi_i^{(mat)}(\kappa_i')$

$$\tilde{\vec{p}}'_j = \vec{p}'_i \text{ and } \tilde{h}'_j = h'_i , \tag{D24}$$

since the momentum and helicity is conserved in the matter, but it has different mass $m_j \neq m_i$ and different 0th order energy

$$\tilde{E}'_j \neq E . \tag{D25}$$

We note that there is only one wavefunction in the expansion of (D22) that satsifies Eq.(D24) with mass m_j. The relation (D24) gives

$$\tilde{\chi}'_j = \chi'_i , \tag{D26}$$

where $\tilde{\chi}'_j$ and χ'_i are the spin spinors of $\phi_j^{(mat)}(\tilde{\kappa}'_j)$ and $\phi_i^{(mat)}(\kappa'_i)$, respectively. The equations (D24) and the substitution rule (A6) yields

$$(\tilde{E}'_j - a_j)^2 = m_j{}^2 + (p'_i - h'_i a_j)^2 \tag{D27}$$

The coefficient β_{ij} is given by

$$\beta_{ij} = \Phi_j{}^{(mat)}(\tilde{\kappa}'_j)^\dagger H_{off} \Phi_i{}^{(mat)}(\kappa'_i) / (E - \tilde{E}'_j). \tag{D28}$$

and it is reduced by Eqs.(D17), (D21) and (D26) to

$$\beta_{ij} = b \cdot \phi_j^{(mat)}(\tilde{\kappa}'_{ij})^\dagger (1 - \gamma^5) \phi_i^{(mat)}(\kappa'_i) / (E - \tilde{E}'_j) . \tag{D28'}$$

$$= \frac{b}{2(E - \tilde{E}'_j)} \sqrt{\frac{(E + m_i - a_i)(\tilde{E}'_j + m_j - a_j)}{(E - a_i)(\tilde{E}'_j - a_j)}} \left(1 - \frac{p'_i h'_i - a_i}{E + m_i - a_i}\right)\left(1 - \frac{p'_i h'_i - a_j}{\tilde{E}'_j + m_j - a_j}\right). \tag{D28''}$$

We express the transmitted wavefunction in the matter as a superposition of energy eigenstates with the same energy and p_x but with different spinors of species,

$$\boldsymbol{\Psi}^{(mat)} = \textstyle\sum_{h'_2=\pm1} T_2(h'_2) \boldsymbol{\Psi}_2{}^{(mat)}(E, \vec{p}'_2, h'_2) + \sum_{h'_1=\pm1} T_1(h'_1) \boldsymbol{\Psi}_1{}^{(mat)}(E, \vec{p}'_1, h'_1) . \tag{D29}$$

where $T_k(h'_k)$ is a transmission coefficient.

D3. Elastic and inelastic reflection rates

D3.1 Condition of continuity of the wave function

We impose the condition that the wave function is continuous at $z = 0$. With Eqs. (D12) and (D29) the condition is written as

$$\begin{aligned}\Psi_2(E, \vec{p}, h) &+ \textstyle\sum_{h''_2=\pm1} R_2(h''_2) \Psi_2(E, \vec{p}''_2, h''_2) + \sum_{h''_1=\pm1} R_1(h''_1) \Psi_1(E, \vec{p}''_1, h''_1) \\ &= \textstyle\sum_{h'_2=\pm1} T_2(h'_2) \boldsymbol{\Psi}_2{}^{(mat)}(E, \vec{p}'_2, h'_2) + \sum_{h'_1=\pm1} T_1(h'_1) \boldsymbol{\Psi}_1{}^{(mat)}(E, \vec{p}'_1, h'_1)\end{aligned} \tag{D30}$$

This gives

$$\Phi_2(E,\vec{p},h) + \sum_{h_2''=\pm1} R_2(h_2'')\Phi_2(E,\vec{p}_2'',h_2'') + \sum_{h_1''=\pm1} R_1(h_1'')\Phi_1(E,\vec{p}_1'',h_1'')$$

$$= \sum_{h_2'=\pm1} T_2(h_2')\boldsymbol{\Phi}_2^{(mat)}(E,\vec{p}_1',h_2') + \sum_{h_1'=\pm1} T_1(h_1')\boldsymbol{\Phi}_1^{(mat)}(E,\vec{p}_1',h_1') \quad \text{(D31)}$$

The equation (D31) gives continuity conditions both on ν_1 and ν_2 components as：

$$\phi_2(E,\vec{p},h_2) + \sum_{h_2''=\pm1} R_2(h_2'')\phi_2(E,\vec{p}_2'',h_2'')$$

$$= \sum_{h_2'=\pm1} T_2(h_2')\phi_2^{(mat)}(E,\vec{p}_2',h_2') + \sum_{h_1'=\pm1} T_1(h_1')\beta_{12}\phi_2^{(mat)}(\tilde{E}_2',\vec{p}_1',h_1') \quad \text{(D32)}$$

and

$$\sum_{h_1''=\pm1} R_1(h_1'')\phi_1(E,\vec{p}_1'',h_1'')$$

$$= \sum_{h_2'=\pm1} T_2(h_2')\beta_{21}\phi_1^{(mat)}(\tilde{E}_1',\vec{p}_2',h_2') + \sum_{h_1'=\pm1} T_1(h_1')\phi_1^{(mat)}(E,\vec{p}_1',h_1'), \quad \text{(D33)}$$

respectively. In order to solve Eqs.(D32) and (D33) we use the perturbation method. We first expand $R_i(h_i'')$ and $T_i(h_i')$ in power of b as

$$R_i(h_i'') = R_i^{(0)}(h_i'') + R_i^{(1)}(h_i'') + \dots \quad \text{(D34)}$$

$$T_i(h_i') = T_i^{(0)}(h_i') + T_i^{(1)}(h_i') + \dots \quad \text{(D35)}$$

and then put

$$R_1^{(0)}(h_1'') = T_1^{(0)}(h_1') = 0\,, \quad \text{(D36)}$$

since there should be no mass changing reflection or transmission in the 0th order. Comparing both sides of Eqs.(D32) and (D33) and taking the terms with the same order in b, we find

$$\phi_2(E,\vec{p},h_2) + \sum_{h_2''=\pm1} R_2^{(0)}(h_2'')\phi_2(E,\vec{p}_2'',h_2'')$$

$$= \sum_{h_2'=\pm1} T_2^{(0)}(h_2')\phi_2^{(mat)}(E,\vec{p}_2',h_2') \quad \text{(D37)}$$

for the 0th order, and

$$\sum_{h_1''=\pm1} R_1^{(1)}(h_1'')\phi_1(E,\vec{p}_1'',h_1'') - \sum_{h_1'=\pm1} T_1^{(1)}(h_1')\phi_1^{(mat)}(E,\vec{p}_1',h_1')$$

$$= \sum_{h_2'=\pm1} T_2^{(0)}(h_2')\beta_{21}\phi_1^{(mat)}(\tilde{E}_1',\vec{p}_2',h_2')\,, \quad \text{(D38)}$$

for the 1st order. The equations (D37) and (D38) should give the elastic reflection coefficient $R_2^{(0)}(h_2'')$ and the inelastic one $R_1^{(1)}(h_1'')$, respectively, in the lowest order.

D3.2 Elastic reflection rate and the total reflection

In order to obtain the inelastic reflection coefficients we need to solve the equation (D37) for elastic coefficients, since the equation (D38) for inelastic coefficients has elastic transmission coefficients $T_2^{(0)}(h_2')$ on the right-hand side. The equation (D37) gives the continuity conditions for the upper and lower components of Dirac spinors,

$$N_2[\chi_2 + \sum_{h_2''=\pm1} R_2^{(0)}(h_2'')\chi_2''(h_2'')] = N_2' \sum_{h_2'=\pm1} T_2^{(0)}(h_2')\chi_2'(h_2'),$$
$$N_2[\frac{ph}{E+m}\chi_2 + \sum_{h_2''=\pm1} R_2^{(0)}(h_2'')\frac{ph_2''}{E+m}\chi_2''(h_2'')] = N_2' \sum_{h_2'=\pm1} T_2^{(0)}(h_2')\frac{p_2'h_2' - a_2}{E - a_2 + m}\chi_2'(h_2') \qquad \text{(D39)}$$

The equations are solved in the same way as in Appendix A.

Let us solve in the following a simple case where the incident neutrino ν_2 is nonrelativistic with

$$|a_i| << p, p_2' << m_2 \approx E. \qquad \text{(D40)}$$

In this case we have approximately $\vec{p}_2'(h_2' = 1) = \vec{p}_2'(h_2' = -1)$, which makes the calculation simple. Using new normalized spinors χ_2'' and χ_2' defined by

$$\sum_{h_2''=\pm1} R_2^{(0)}(h_2'')\chi_2''(h_2'') = \tilde{R}_2^{(0)}\chi_2'' \qquad \text{with } |\tilde{R}_2^{(0)}|^2 = |R_2^{(0)}(+1)|^2 + |R_2^{(0)}(-1)|^2$$
$$\sum_{h_2'=\pm1} T_2^{(0)}(h_2')\chi_2'(h_2') = \tilde{T}_2^{(0)}\chi_2' \qquad \text{with } |\tilde{T}_2^{(0)}|^2 = |T_2^{(0)}(+1)|^2 + |T_2^{(0)}(-1)|^2 \qquad \text{(D41)}$$

we get from Eq.(D41) in the nonrelativistic limit of the incident neutrino ν_2,

$$\sum_{h_2''=\pm1} R_2^{(0)}(h_2'')ph''\chi_2''(h_2'') = \tilde{R}_2^{(0)}\vec{\sigma}\vec{p}_2''\chi_2''$$
$$\sum_{h_2'=\pm1} T_2^{(0)}(h_2')p_2'h_2'\chi_2'(h_2') = \vec{\sigma}\vec{p}_2'\tilde{T}_2^{(0)}\chi_2' \qquad \text{(D42)}$$

Then using Eqs.(D41) we rewrite Eqs.(D39) as

$$\chi_2 + \tilde{R}_2^{(0)}\chi_2'' = \tilde{T}_2^{(0)}\chi_2'$$
$$\vec{\sigma}\vec{p}\chi_2 + \vec{\sigma}\vec{p}_2''\tilde{R}_2^{(0)}\chi_2'' \approx \vec{\sigma}\vec{p}_2'\tilde{T}_2^{(0)}\chi_2' \qquad \text{(D43)}$$

Eliminating χ_2' from Eqs.(D43), we get the equation

$$\vec{\sigma}(\vec{p} - \vec{p}_2')\chi_2 + \tilde{R}_2^{(0)}(\vec{\sigma}(\vec{p}_2'' - \vec{p}_2'))\chi_2'' = 0 \qquad \text{(D44)}$$

Taking into account the conservation of x-component of momentum p_x, we obtain

$$\sigma_z(p_z - (p_2')_z)\chi_2 + \tilde{R}_2^{(0)}\sigma_z(-p_z - (p_2')_z)\chi_2'' = 0, \quad \therefore \tilde{R}_2^{(0)}\chi_2'' = \frac{p_z - (p_2')_z}{p_z + (p_2')_z}\chi_2$$

From this we obtain, apart form phases,

$$\tilde{R}_2^{(0)} = \frac{p_z - (p_2')_z}{p_z + (p_2')_z} \qquad \text{(D45)}$$

and

$$\chi_2'' = \chi_2. \qquad \text{(D46)}$$

The equation (D45) can be rewritten as

$$\tilde{R}_2^{(0)} = \frac{\cos\alpha - n\cos\alpha'}{\cos\alpha + n\cos\alpha'} = \frac{\sin\theta - \sqrt{\sin^2\theta - (1-n^2)}}{\sin\theta + \sqrt{\sin^2\theta - (1-n^2)}} \qquad \text{(D47)}$$

with the refraction index n, the angle of incidence α, the grazing angle $\theta = \pi/2 - \alpha$ and the angle of refraction α' (see Fig.D1). Similarly we obtain the transmission coefficient $\tilde{T}_2^{(0)}$

$$\tilde{T}_2^{(0)} = \frac{p_z - (p_2'')_z}{(p_2')_z - (p_2'')_z} = \frac{2p_z}{(p_2')_z + p_z} \tag{D48}$$

and

$$\chi_2' = \chi_2 . \tag{D49}$$

The equation (D48) can be rewritten as

$$\tilde{T}_2^{(0)} = \frac{2\cos\alpha}{n\cos\alpha' + \cos\alpha} = \frac{2\sin\theta}{\sin\theta + \sqrt{\sin^2\theta - (1-n^2)}} \tag{D50}$$

with use of the conservation of p_x and the relation

$$n^2 = 1 - 2m_2 a_2 / p^2$$

given by Eq.(8'). We see from Eqs.(D46) and (D49) that the spinor (or equivalently the spin) is conserved in the nonrelativistic process of reflection or transmission. The reflection rate is given by

$$r(2 \to 2) = | R_2^{(0)} |^2 = \left| \frac{\sin\theta - \sqrt{\sin^2\theta - 2m_2 a_2 / p^2}}{\sin\theta + \sqrt{\sin^2\theta - 2m_2 a_2 / p^2}} \right|^2 \tag{D51}$$

We have the total reflection $r(2 \to 2) = 1$ for

$$\sin^2\theta - 2m_2 a_2 / p^2 < 0 , \tag{D52}$$

since the inequality (D52) makes $\sqrt{\sin^2\theta - 2m_2 a_2 / p^2}$ pure imaginary. The transmission rate $t(2 \to 2)$ is given by

$$t(2 \to 2) = \left((v_2')_z / (v_2)_z \right) | \tilde{T}_2^{(0)} |^2 \approx \left([\mathrm{Re}(p_2')_z] / p_z \right) | \tilde{T}_2^{(0)} |^2 \tag{D53}$$

$$= \frac{4\sin\theta \,\mathrm{Re}\sqrt{\sin^2\theta - 2m_2 a_2 / p^2}}{| \sin\theta + \sqrt{\sin^2\theta - 2m_2 a_2 / p^2} |^2} \tag{D53'}$$

Here the velocity dependent factor in Eq.(D53) is due to the definition of the transmission rate; it is the ratio of the flux (namely, the density times the velocity) in z-direction of the transmitted wave and that of the incident wave. We see the equations (D52) and (D53') give the conservation of neutrino flux in the 0th order of b,

$$1 = r(2 \to 2) + t(2 \to 2) \tag{D54}$$

D3.2 Inelastic reflection rate

Next we solve Eq.(38) to obtain the rate of the inelastic reflection $\nu_2 \to \nu_1$. The wave functions appearing in Eq.(38) are given by Eqs.(D10) and (D15). We insert them in Eq. (D38) and compare the upper and lower components of Dirac spinors in both sides of the equation to obtain

$$N_1 \sum_{h_1''=\pm 1} R_1^{(1)}(h_1'')\chi_1''(h_1'') - N_1' \sum_{h_1'=\pm 1} T_1^{(1)}(h_1')\chi_1'(h_1') = \sum_{h_2'=\pm 1} T_2^{(0)}(h_2')\tilde{N}_1'\beta_{21}(h_2')\chi_2'(h_2') \qquad \text{(D55)}$$

and

$$N_1 \sum_{h_1''=\pm 1} R_1^{(1)}(h_1'')\frac{p_1''h_1''}{E+m_1}\chi_1''(h_1'') - N_1' \sum_{h_1'=\pm 1} T_1^{(1)}(h_1')\frac{p_1'h' - a_1}{E-a+m_1}\chi_1'(h_1')$$
$$= \sum_{h_2'=\pm 1} T_2^{(0)}(h_2')\tilde{N}_1'\beta_{21}(h_2')\frac{p_2'h_2' - a_1}{\tilde{E}_1' - a + m_1}\chi_2'(h_2') \qquad \text{(D56)}$$

with $N_1 = \sqrt{\dfrac{E+m_1}{2E}}$, $N_1' = \sqrt{\dfrac{E+m_1-a_1}{2(E-a_1)}}$, $\tilde{N}_1' = \sqrt{\dfrac{\tilde{E}_1'+m_1-a_1}{2(\tilde{E}_1'-a_1)}}$.

We have energy momentum relations

$$\begin{aligned} E^2 &= m_1^2 + p_1''^2 = m_2^2 + p^2, \\ (E-a_1)^2 &= m_1^2 + (p_1' - h_1'a_1)^2, \\ (E-a_2)^2 &= m_2^2 + (p_2' - h_2'a_2)^2, \\ (\tilde{E}_1' - a_1)^2 &= m_1^2 + (p_2' - h_2'a_1)^2 \end{aligned} \qquad \text{(D57)}$$

We note that $\tilde{N}_1'$ depends implicitly on h_2' through the dependence of $\tilde{E}_1'$ on h_2' shown in the last equation of Eqs.(57).

Let us consider the simple case of nonrelativistic ν_2 with $|a_i| << p << m_2$ and massless ν_1 with helicity $h_1'' = h_1' = -1$ (the extension to the general case is straightforward). We have then $N_1 = N_1' = \tilde{N}_1' = \sqrt{1/2}$, and rewrite Eq.(D55) as

$$R_1^{(1)}(-1)\chi_1''(-1) - T_1^{(1)}(-1)\chi_1'(-1) = T_2^{(0)}(-1)\beta_{21}(-1)\chi_2'(-1) \qquad \text{(D58)}$$

We see that Eq.(D56) is equivalent to Eq.(D55) in this case; it is because the wave functions of ν_1 are eigenstates of γ^5 with $\gamma^5 = -1$. The relation (D57) is reduced to

$$\begin{aligned} E &= m_2 = p_1'' = p_1' \\ p_2'^2 &= p^2 - 2m_2 a_2 \ , \\ \tilde{E}_1' &= p_2' \end{aligned} \qquad \text{(D57')}$$

and Eq. (D28") yields

$$\beta_{21}(-1) = \frac{\sqrt{2}b}{m_2}. \tag{D59}$$

Multiplying both sides of Eq.(D58) with $\chi_1'^{\dagger}(+1)$ we obtain the reflection rate as

$$R_1^{(1)}(-1) = (\sqrt{2}b / m_2) \cdot T_2^{(0)}(-1) \cdot [\chi_1'^{\dagger}(+1)\chi_2'(-1) / \chi_1'^{\dagger}(+1)\chi_1''(-1)] \tag{D60}$$

We note Eqs. (D40) and (D48) give

$$T_2^{(0)}(-1) = \chi_2^{\dagger}(-1)\chi_2(h)\tilde{T}_2^{(0)}, \tag{D61}$$

where h is the helicity of the incident ν_2. When two spinors ξ_1 and ξ_2 satisfy $\vec{\sigma}\hat{n}_i\xi_i = \lambda_i\xi_i$ ($\lambda_i = \pm 1$) with unit vectors $\hat{n}_i$ ($i = 1,2$), the product of them is given (see Appendix C), apart from a phase, as

$$\xi_1^{\dagger}(\lambda_1)\xi_2(\lambda_2) = \sqrt{(1 + \lambda_1\lambda_2\hat{n}_1\hat{n}_2)/2} \tag{D62}$$

We define unit vectors as $\hat{n}_1' \equiv \vec{p}_1'/p_1'$, $\hat{n}_1'' \equiv \vec{p}_1''/p_1''$, $\hat{n}_2 \equiv \vec{p}_2/p_2$, and $\hat{n}_2' \equiv \vec{p}_2'/p_2'$. Using Eqs.(D60), (D61) and (D62) we obtain

$$R_1^{(1)}(-1) = \frac{\sqrt{2}b}{m_2} \cdot \sqrt{\frac{(1 - h_2\hat{n}_2'\hat{n}_2)(1 - \hat{n}_1'\hat{n}_2')}{2(1 - \hat{n}_1'\hat{n}_1'')}} \cdot \tilde{T}_2^{(0)} \tag{D63}$$

We note that $\vec{p}_1''$ the momenta of the reflected ν_1 and $\vec{p}_1'$ that of the transmitted ν_1 are almost in the z-direction, owing to $p_1'', p_1' >> |(p_1'')_x|, |(p_1')_x$, which is derived by $(p_1'')_x = (p_1')_x = p_x$ due to the conservation of the transverse momentum and $p_1'', p_1' = m_2 >> p$ due to the relations (D40) and (D56'). We therefore put

$$\hat{n}_1' = (0,0,1), \quad \hat{n}_1'' = (0,0,-1), \quad \hat{n}_2 = (\sin\alpha, 0, \cos\alpha), \quad \hat{n}_2' = (\sin\alpha', 0, \cos\alpha') \tag{D64}$$

With Eqs.(D50), (D63) and (D64) we have

$$R_1^{(1)}(-1) = \sqrt{2}bm_2^{-1}\frac{\sqrt{(1-\cos\alpha')(1 - h_2\cos(\alpha - \alpha'))}\cos\alpha}{(n\cos\alpha' + \cos\alpha)} \tag{D65}$$

We obtain the inelastic reflection rate

$$\begin{aligned} r(2 \to 1) &= |(v_1'')_z/(v_2)_z| \cdot |R_1^{(1)}|^2 = (m_2/p\cos\alpha)\cdot |R_1^{(1)}|^2 \\ &= \frac{2b^2}{pm_2} \cdot \frac{\cos\alpha(1-\cos\alpha')(1 - h_2\cos(\alpha' - \alpha))}{(n\cos\alpha' + \cos\alpha)^2} \end{aligned} \tag{D66}$$

for the nonrelativistic incident neutrino ν_2 and the massless reflected meutrino ν_1. The factor p^{-1} on the right-hand side is inherent to the rate of exoergic reaction process. Similarly we have the inelastic transmission rate,

$$t(2 \to 1) = \frac{2b^2}{pm_2} \cdot \frac{\cos\alpha(1+\cos\alpha')(1 - h_2\cos(\alpha' - \alpha))}{(n\cos\alpha' + \cos\alpha)^2} \tag{D66'}$$

Let us compare the inelastic reflection rate Eq.(D66) with the elastic reflection rate Eq.(D51). The former has a factor b^2 / pm_2, while the latter has a factor $|n^2 - 1|^2 \sim (2m_2 a_2 / p^2)^2$, and the ratio of these factors is $\sim (p / m_2)^3 \sim (T / m_2)^3$ and very small, since $b \sim a_2$ in the ordinary weak potential. Although the inelastic reflection yields a large recoil momentum $p_1'' \sim m_2$ rather than $p \sim T$ in the elastic case, the smallness of the inelastic reflection rate gives the net effective ratio of momentum transfer $\sim (p / m_2)^2 \sim (T / m_2)^2$, which is very small, say, much less than 10^{-3}. Moreover the elastic reflection is much enhanced for small grazing angle, while the inelatic reflection does not have such enhancement. In conclusion we find

(i) the inelastic reflection is smaller than the elastic reflection by many orders of magnitude when the total reflection is realized in the 0th order perturbation,

(ii) the inelastic reflection rate is proportional to b^2 / pm_2, namely, to the square of the off-diagonal potential strength and to the inverse of the incident momentum and also to the inverse of the mass difference (in case of $m_1 = 0$, it is m_2),

(iii) the inelastic reflection rate is explicitly given; the result of our 2-species model is straightforwardly extended to 3-species case, giving the rate for the reflection $\nu_i \to \nu_j$

$$r(i \to j) = \frac{2V_*^2 \,|\, W_{ei} W_{ej}^* \,|^2}{pm_i} \cdot \frac{\cos\alpha (1 - \cos\alpha')(1 - h\cos(\alpha' - \alpha))}{(n\cos\alpha' + \cos\alpha)^2} \quad , \qquad \text{(D67)}$$

with $V_* = (G_F / \sqrt{2})\rho_p$ and W_{ei} given by Eqs.(42) and (45), when the incident neutrino ν_i with mass m_i and helicity h is nonrelativistic and the reflected neutrino ν_j is massless. We also have the inelastic transmission rate for the 3-species case,

$$t(i \to j) = \frac{2V_*^2 \,|\, W_{ei} W_{ej}^* \,|^2}{pm_i} \cdot \frac{\cos\alpha (1 + \cos\alpha')(1 - h\cos(\alpha' - \alpha))}{(n\cos\alpha' + \cos\alpha)^2} \qquad \text{(D67')}$$

(iv) the inelastic reflection produces a large recoil momentum $\sim \sqrt{|\, m_i^2 - m_j^2 \,|}$, but the reaction rate is very small, by a factor $\sim (T / m_i)^3$ compared with the elastic reflection rate. Taking into account the enhancement of the elastic reflection at small grazing angle, the ratio of the reflection rate is much less.