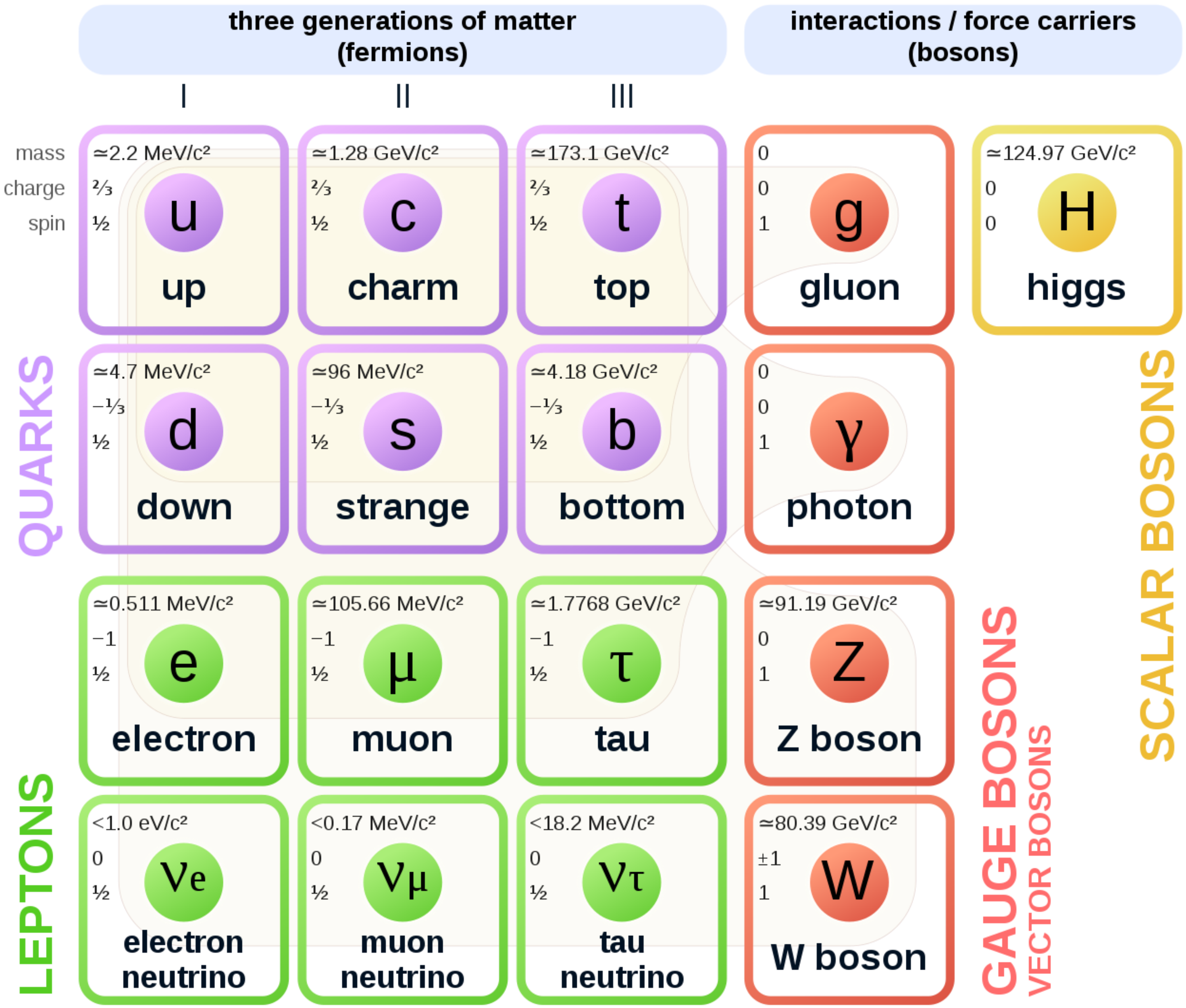
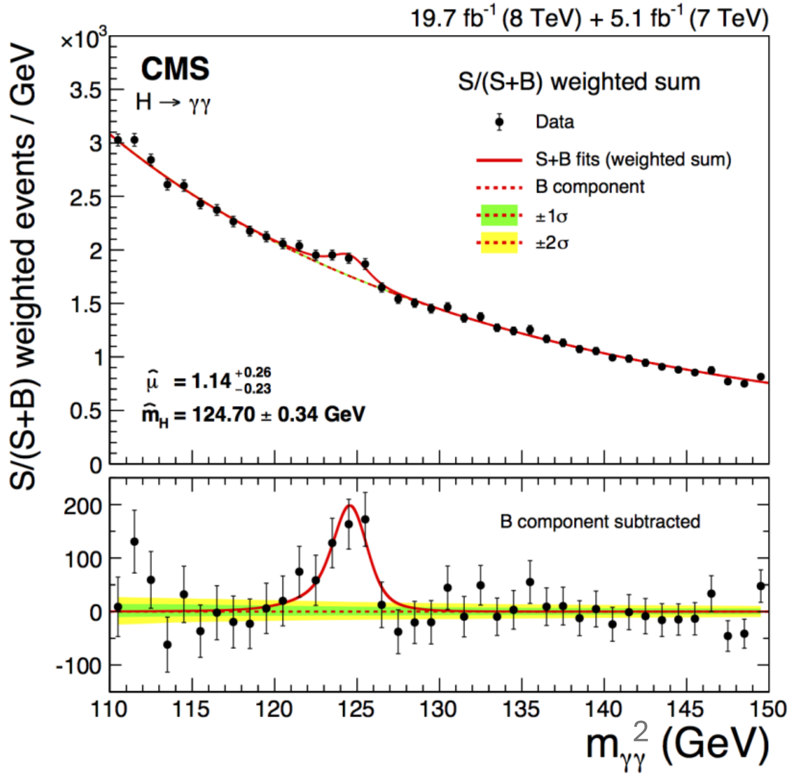
# Quantum dark optics of axions: how can dark matter axion generate quantized photons?

Akira Miyazaki

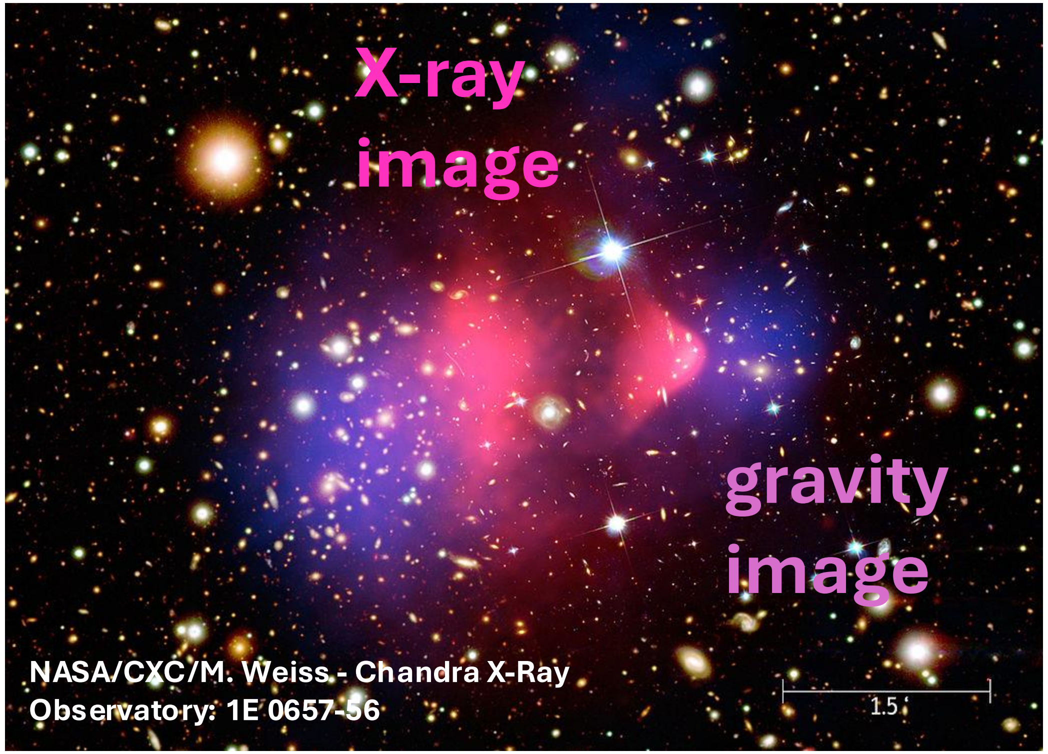
From wikipedia

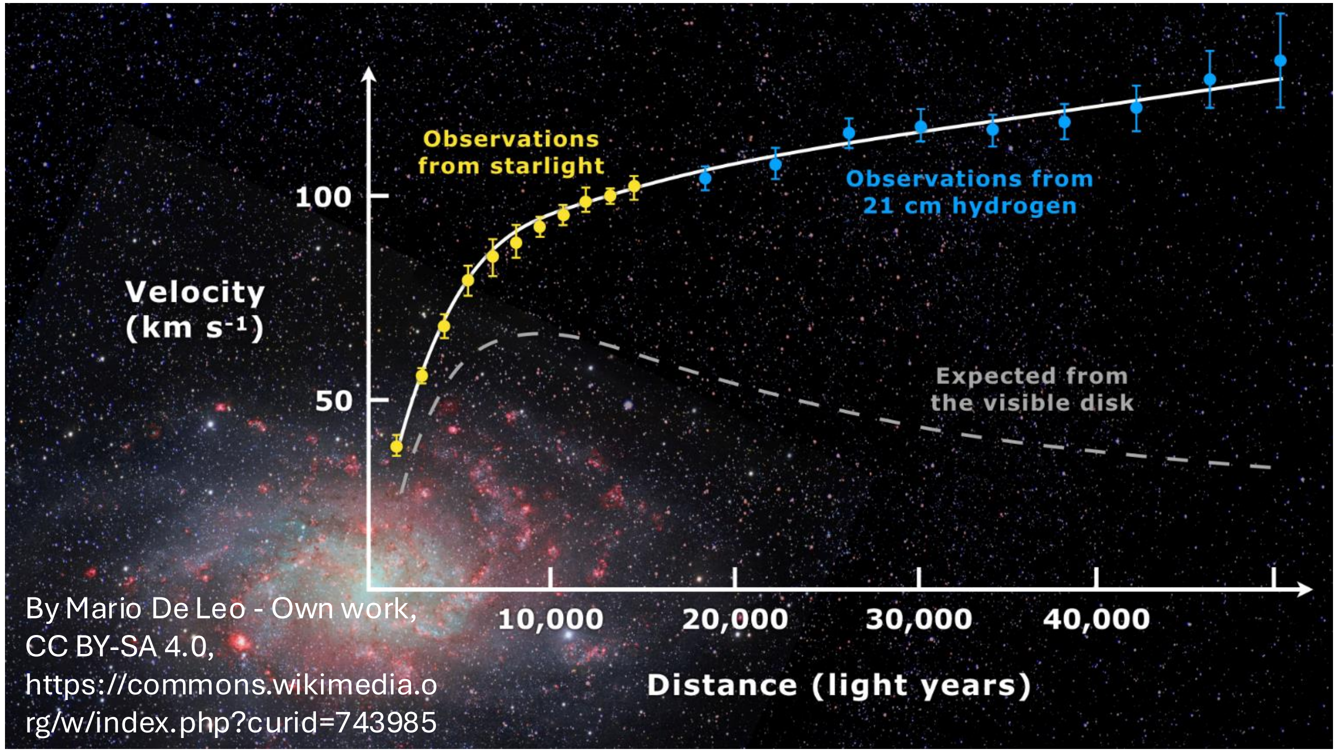
# Standard Model of Elementary Particles



# Clear need for new physics: e.g. Dark Matter (DM)




~~Neutrino~~?

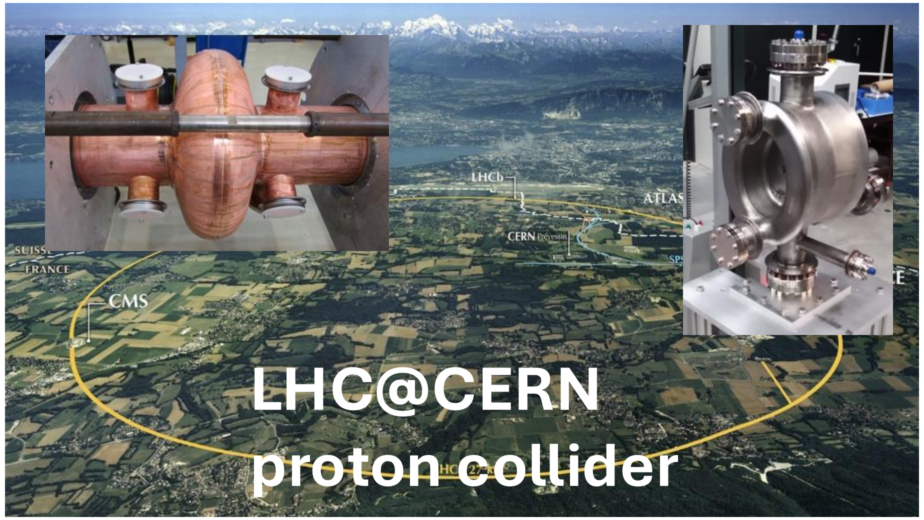
~~Dark astrophysical objects~~?

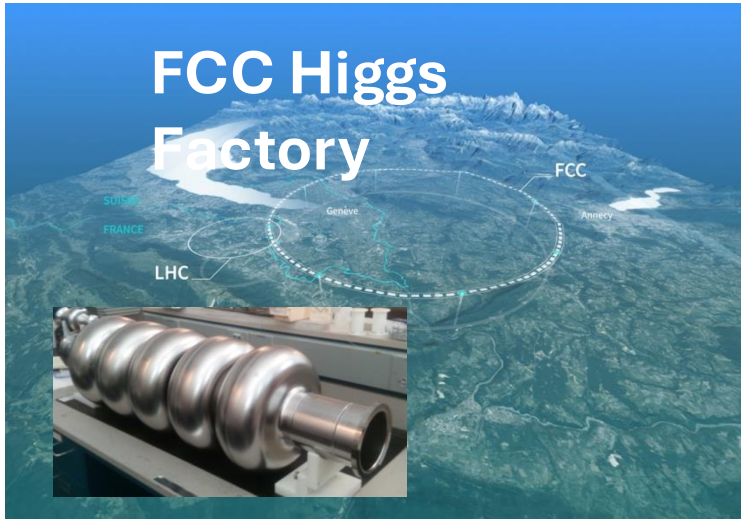
~~Modified gravity~~? van Dokkum, *et al. Nature* **555,** 629–632 (2018)

Primordial black holes?

✓ Hypothetical new particles linked to intrinsic issues in the Standard Model?

# My main business: Radio Frequency cavities for accelerators

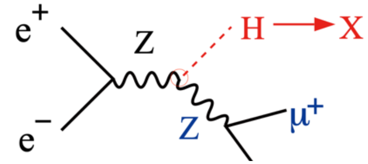



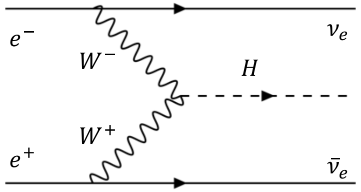


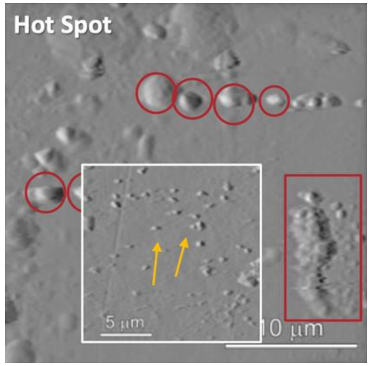


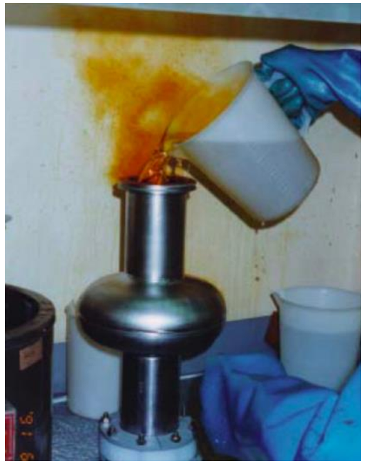
Pros: any present and future accelerators would include superconducting RF cavities
Cons: Timeline (decades?) costs (>>BEUR?) still no promise for ***new physics***

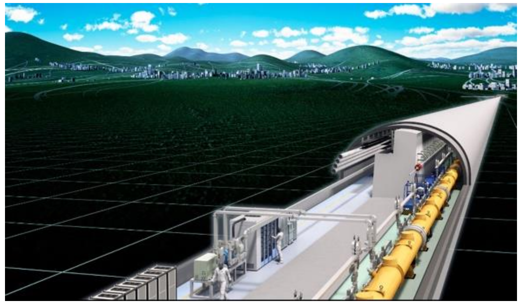
# Road map for the NEXT colliders

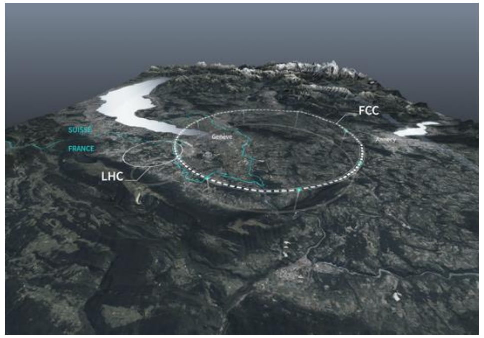


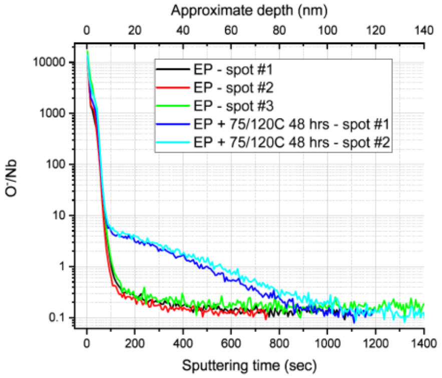



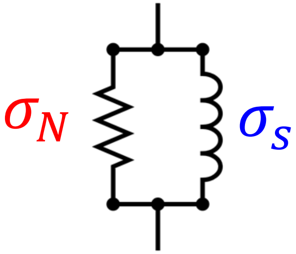



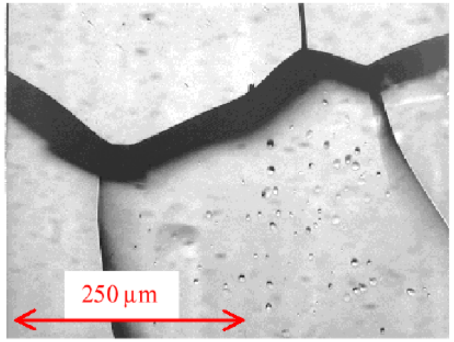



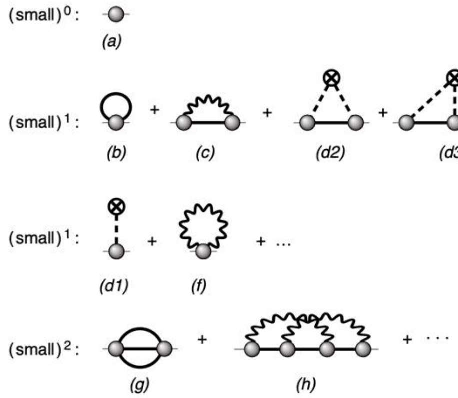




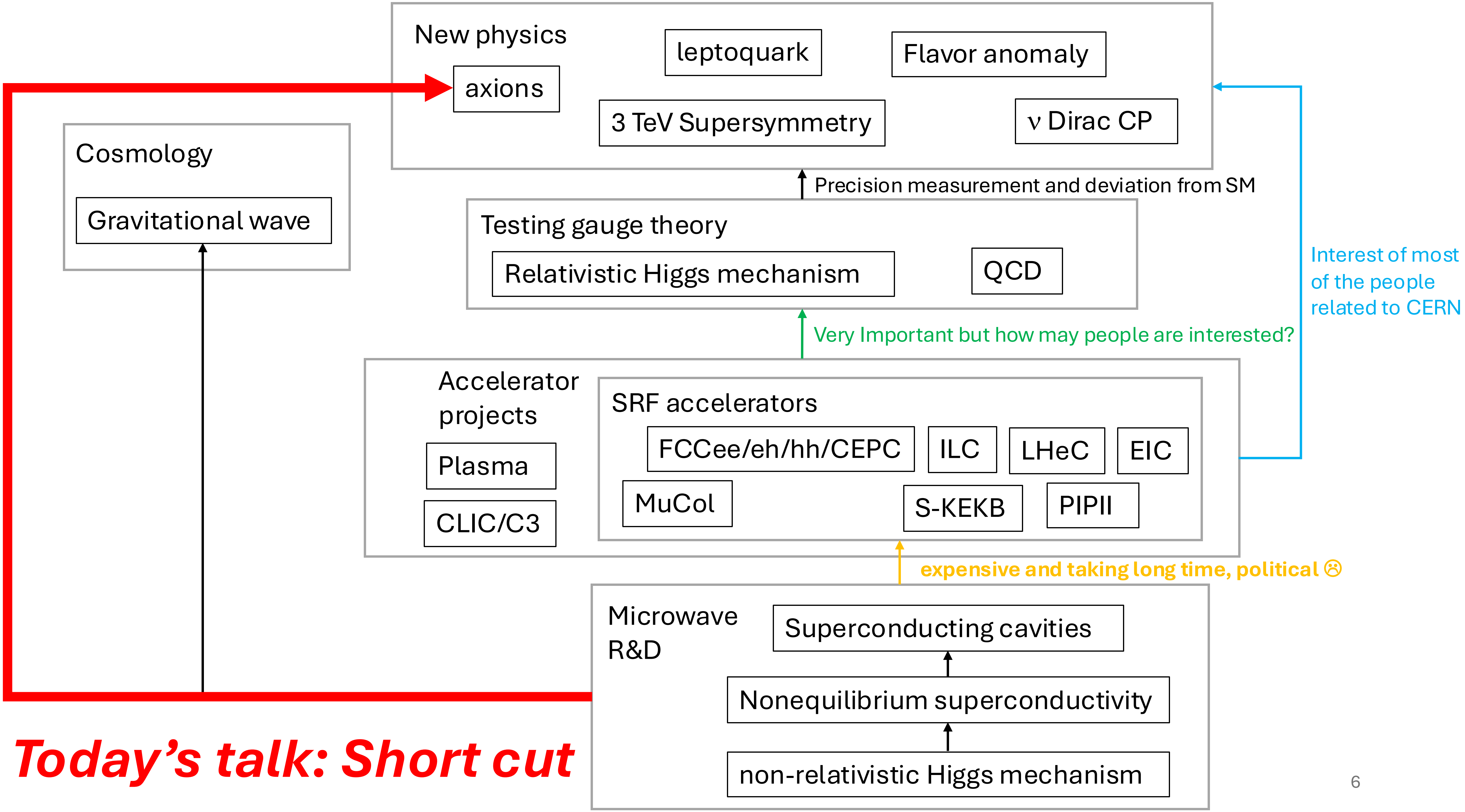




$$M\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \eta\frac{\partial \boldsymbol{u}}{\partial t} - \epsilon\frac{\partial^2 \boldsymbol{u}}{\partial z^2} + \nabla U(z,u) = \boldsymbol{J}_{RF}(z,u) \times \boldsymbol{B}_{\mathrm{ext}}$$

Highly performing superconducting RF cavities for the precision measurement of Higgs boson


New physics
axions
leptoquark
Flavor anomaly
3 TeV Supersymmetry
ν Dirac CP
Cosmology
Gravitational wave
Precision measurement and deviation from SM
Testing gauge theory
Relativistic Higgs mechanism
QCD
Interest of most of the people related to CERN
Very Important but how may people are interested?
Accelerator projects
Plasma
CLIC/C3
SRF accelerators
FCCee/eh/hh/CEPC
ILC
LHeC
EIC
MuCol
S-KEKB
PIPII
expensive and taking long time, political ☹
Microwave R&D
Superconducting cavities
Nonequilibrium superconductivity
non-relativistic Higgs mechanism
Today's talk: Short cut

# Outline

- Background: classical to quantum detection of axions
- Motivation of this talk: classical to quantum??
- Rigorous proof of classical to quantum
- Applications: coherence & homodyne with polarization
- Conclusion

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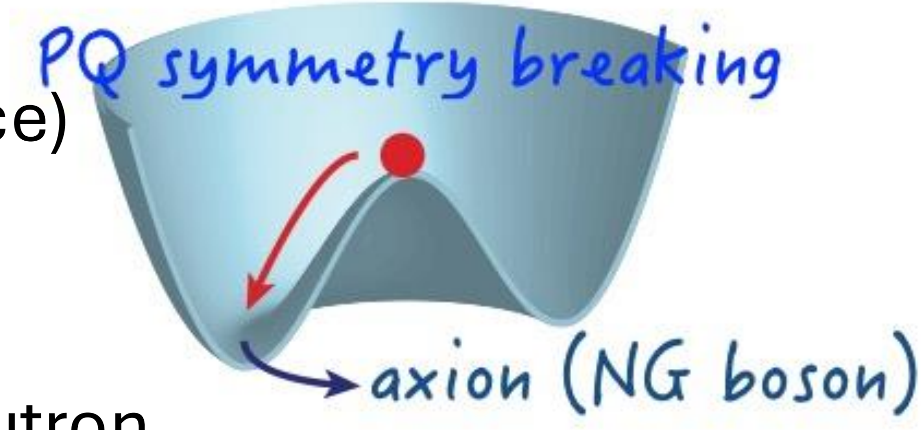
# Axions: a byproduct to cancel the strong CP




Quantum Chromodynamics (theory of strong force)

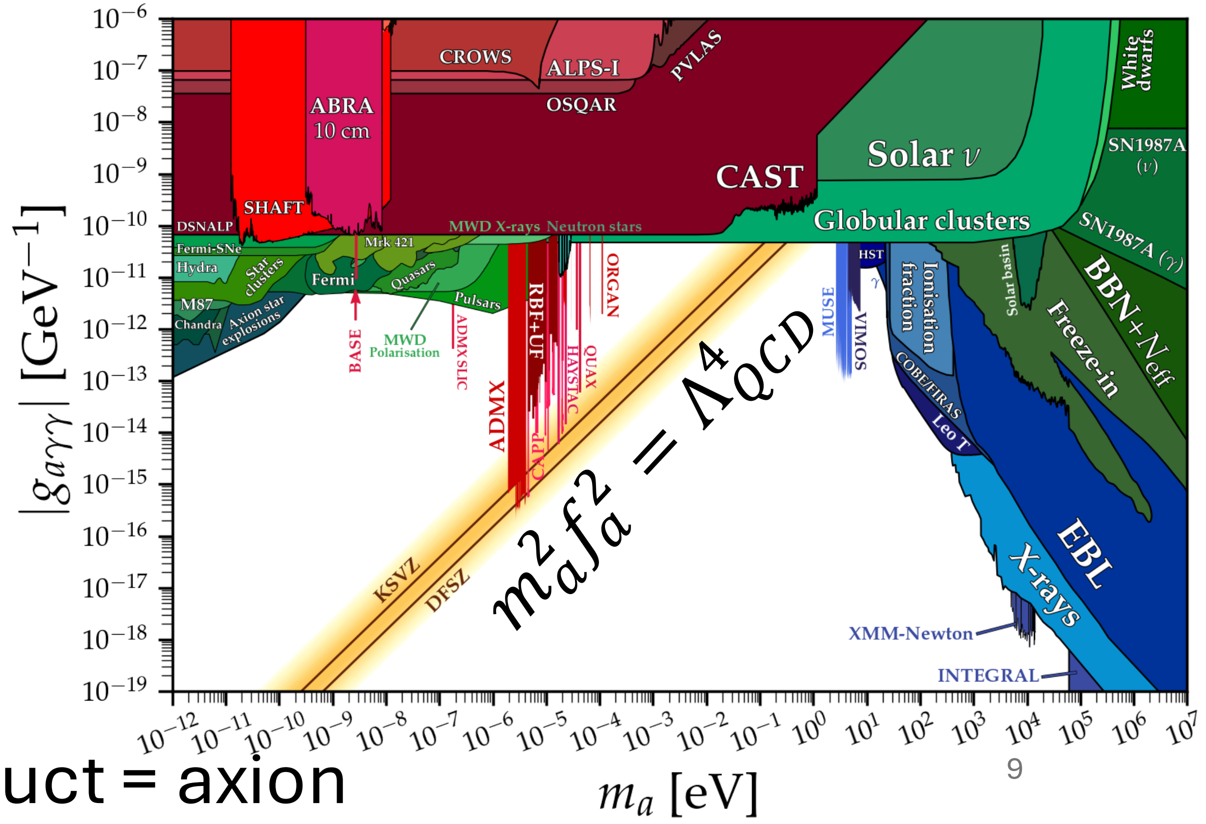
gluons

$$L_{QCD} \supset -\frac{1}{4} G^{a}_{\mu\nu} G^{\mu\nu a} + \frac{g_s^2}{32\pi^2} \boldsymbol{\theta} G^{a}_{\mu\nu} \tilde{G}^{\mu\nu a}$$

This term generates electric dipole moment in neutron

- Theory: $d_n \sim 4.5 \times 10^{-15} \boldsymbol{\theta}$ ecm
- Experiment: $|d_n| < 2.9 \times 10^{-29}$ ecm

$\rightarrow |\theta| < 0.7 \times 10^{-11} \ll 1$




***Naturalness without anthropic solution***

Introduce a new global chiral U(1) field $a$

$$\frac{g_s^2}{32\pi^2}\left(\theta + \frac{\boldsymbol{a}}{F_a}\right) G^{a}_{\mu\nu} \tilde{G}^{\mu\nu a} \rightarrow 0 \text{ (after SSB)}$$

SSB→ A pNG boson appears as a byproduct = axion

# Axion gains mass from QCD $m_a f_a \sim m_\pi f_\pi$

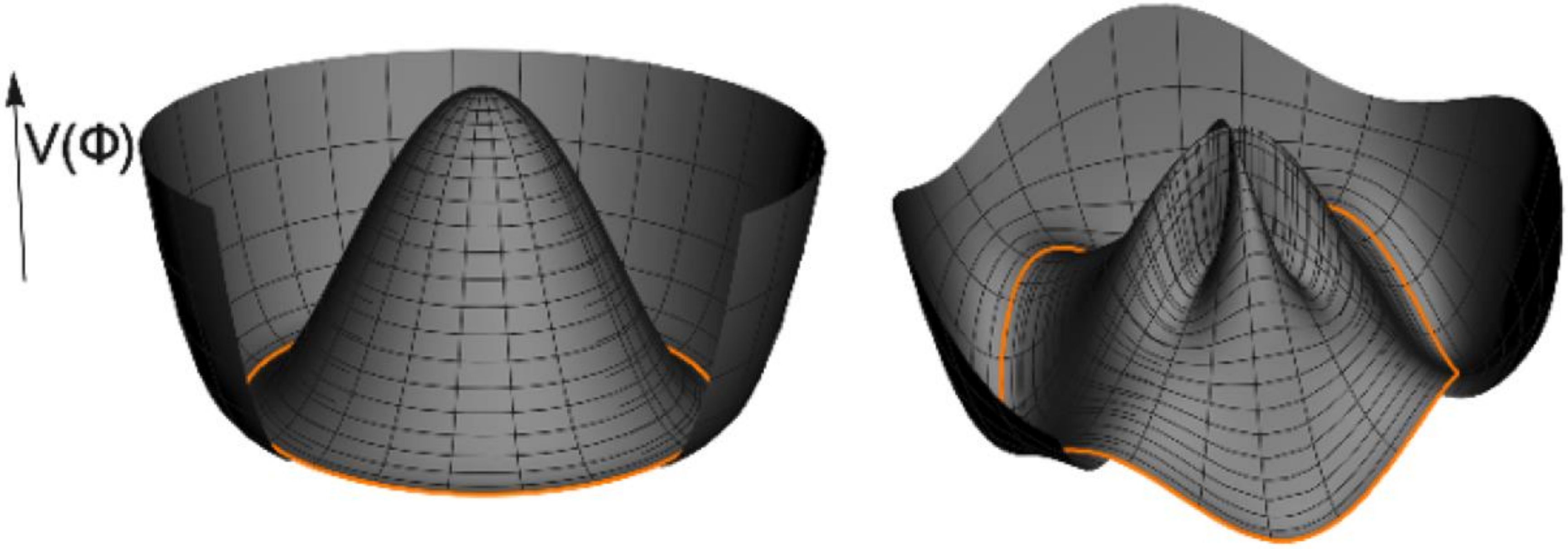
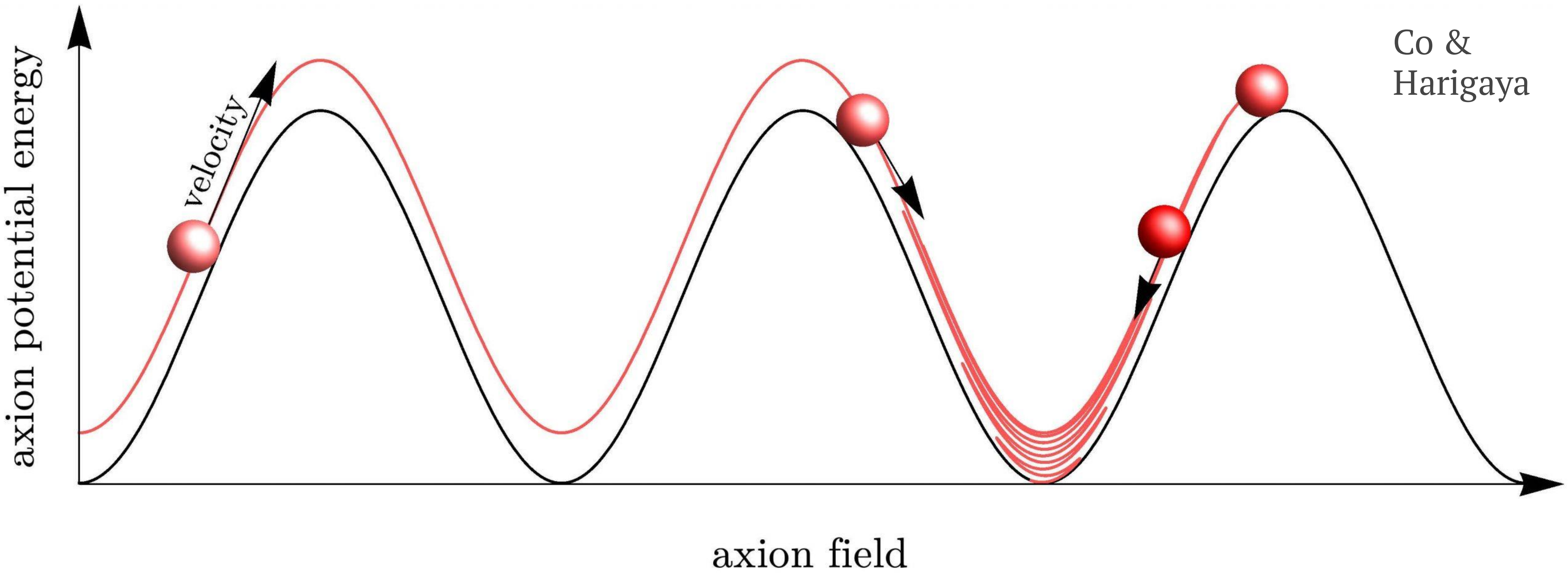



Figure 3: “Sombrero” potential of the Peccei-Quinn field Φ is shown schematically before (left) and after (right) the QCD phase transition. The axion corresponds to the angular direction of this potential, shown by the orange line. The state of the field is given by a point in the potential. Low energy configurations are favoured. For illustration, the potential on the right is shown for a scenario with a large amount of PQ symmetry breaking. More details are given in appendix A.2.

F. Chadha-Day, J. Ellis, D. J. E. Marsh, ”Axion Dark Matter: What is it and Why Now?” arXiv:2105.01406

# Axion as dark matter (PQ scale > inflation)

Axion loses kinetic energy ***non-thermally*** by coherent oscillation in the PQ potential



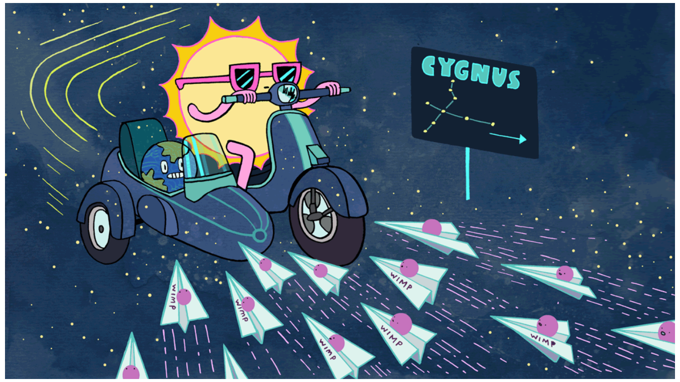

Misalignment mechanism

$$\frac{d^2a}{dt^2} + 3\boxed{H(t)}\frac{da}{dt} + m_a(T)^2 a = 0$$

Hubble constant: friction → cooling down

# De Broglie wavelength $\lambda_B$ vs density of DM $\bar{n}$

- We are moving in the **galaxy halo of dark matter** with speed of 220 km/s → $\beta \sim 0.07\%$
- **De Broglie wavelength** $\lambda_B = 196 \text{ MeVfm}/mv$
- **Galaxy halo of dark matter density** $\rho \sim 0.4 \ \text{GeV/cm}^3$




https://www.symmetrymagazine.org/article/wimps-in-the-dark-matter-wind Artwork by Sandbox Studio, Chicago with Corinne Mucha

<u>WIMP: $m \sim 1$ TeV (?)</u>

$$\lambda_B \sim \frac{196 \text{ MeVfm}}{0.7 \text{ GeV}} = 0.3 \text{ fm}$$

$$\bar{n} \sim \frac{0.4 \ \text{GeV/cm}^3}{1 \text{ TeV}} \sim 10^{-3} \text{ cm}^{-3}$$

$$\rightarrow \bar{n} \ll \frac{1}{\lambda^3}$$

WIMP behaves as a particle

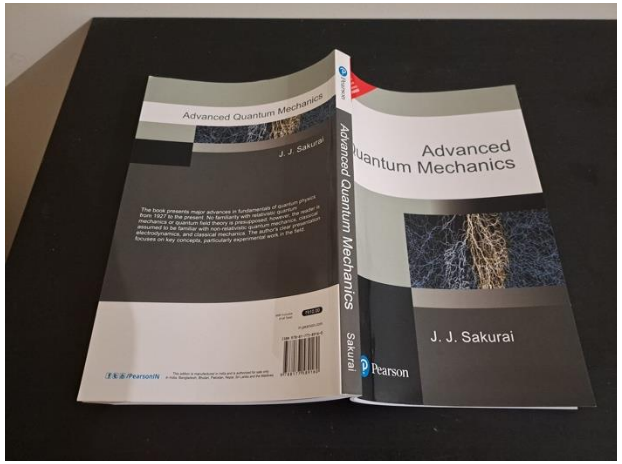
<u>Axions: $m \sim 10 \ \mu\text{eV}$</u>

$$\lambda_B \sim \frac{2 \times 10^{-7} \text{ eVm}}{7 \text{ neV}} = 28 \text{ cm}$$

$$\bar{n} \sim \frac{0.4 \ \text{GeV/cm}^3}{10 \ \mu\text{eV}} \sim 10^{13} \text{ cm}^{-3}$$

$$\rightarrow \bar{n} \gg \frac{1}{\lambda^3}$$

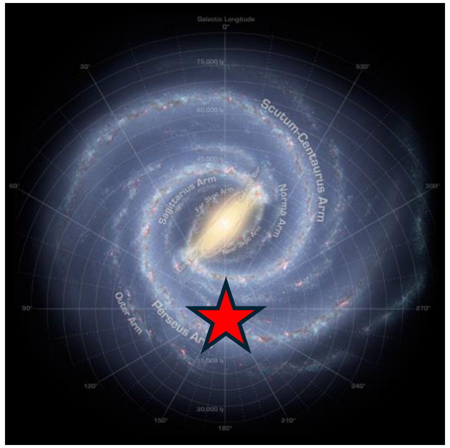
***DM Axions behave as a wave***

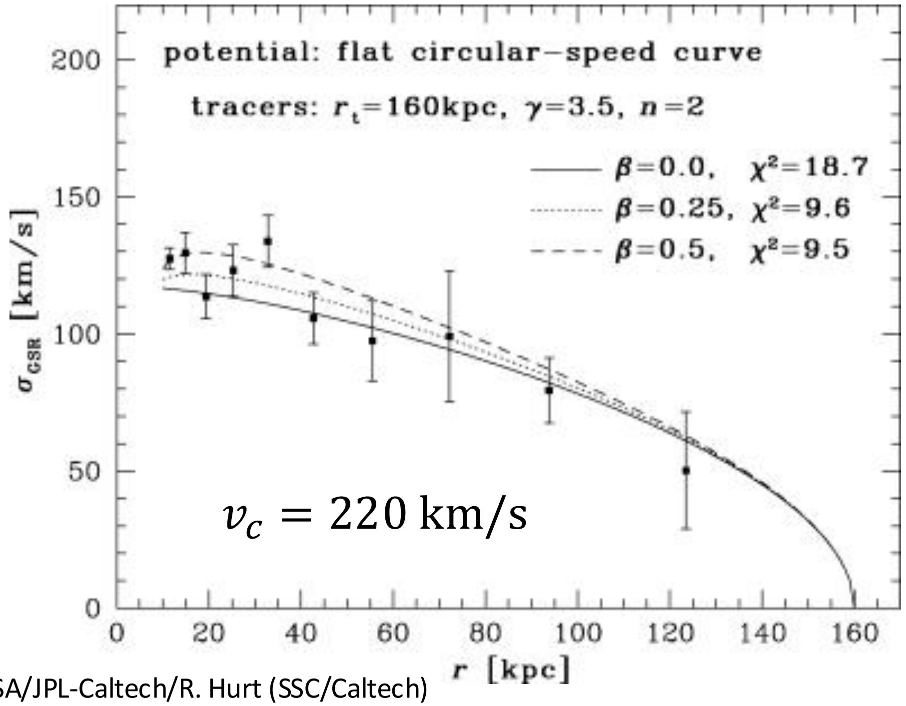



$$E_{cl}^2 = \frac{\bar{n}}{\lambda} \gg \frac{1}{\lambda^4} \sim \langle 0 | \hat{E} \cdot \hat{E} | 0 \rangle$$

# Standard model of dark matter axion distribution function

Velocity dispersion of Milky way Galaxy around



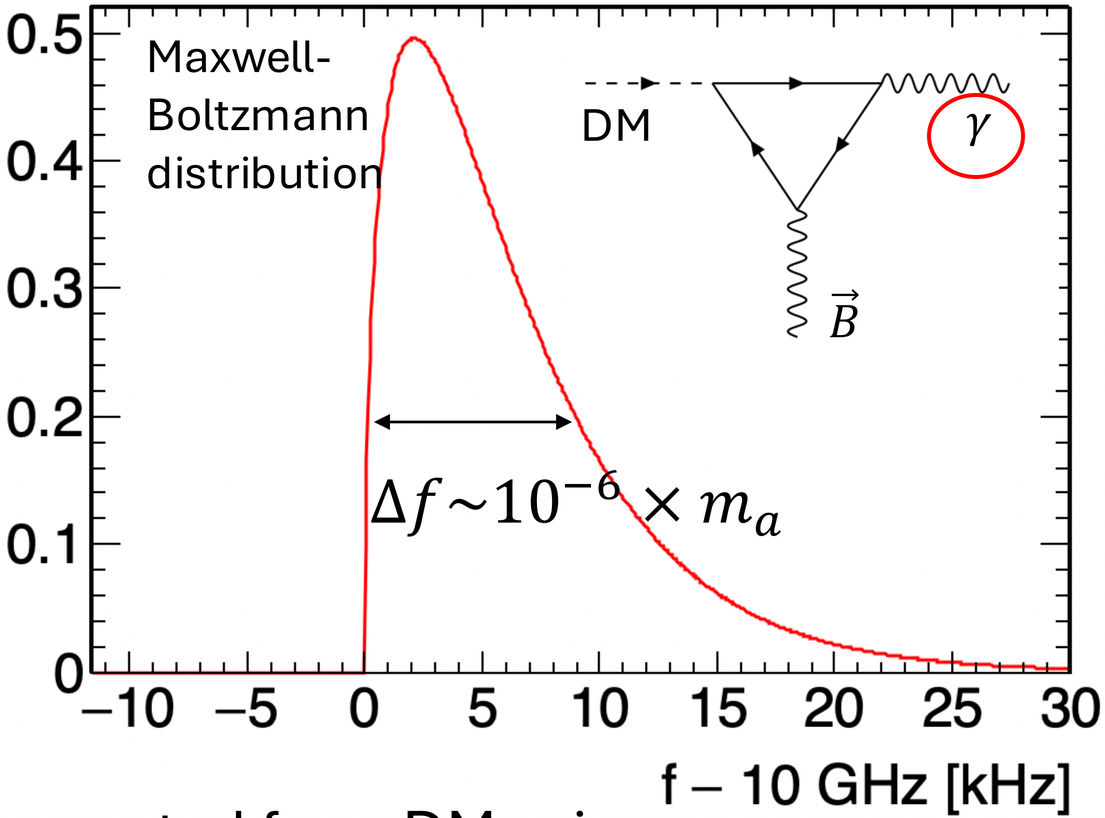



NASA/JPL-Caltech/R. Hurt (SSC/Caltech)

*Mon Not R Astron Soc*, Volume 369, Issue 4, July 2006, Pages 1688–1692,

$$f(\omega) = P_0\theta(\omega - m_a)2\omega_0^{-\frac{3}{2}}\sqrt{\frac{\omega - m_a}{\pi}}\exp\left(-\frac{\omega - m_a}{\omega_0}\right)$$

If $m_a = 10$ GHz



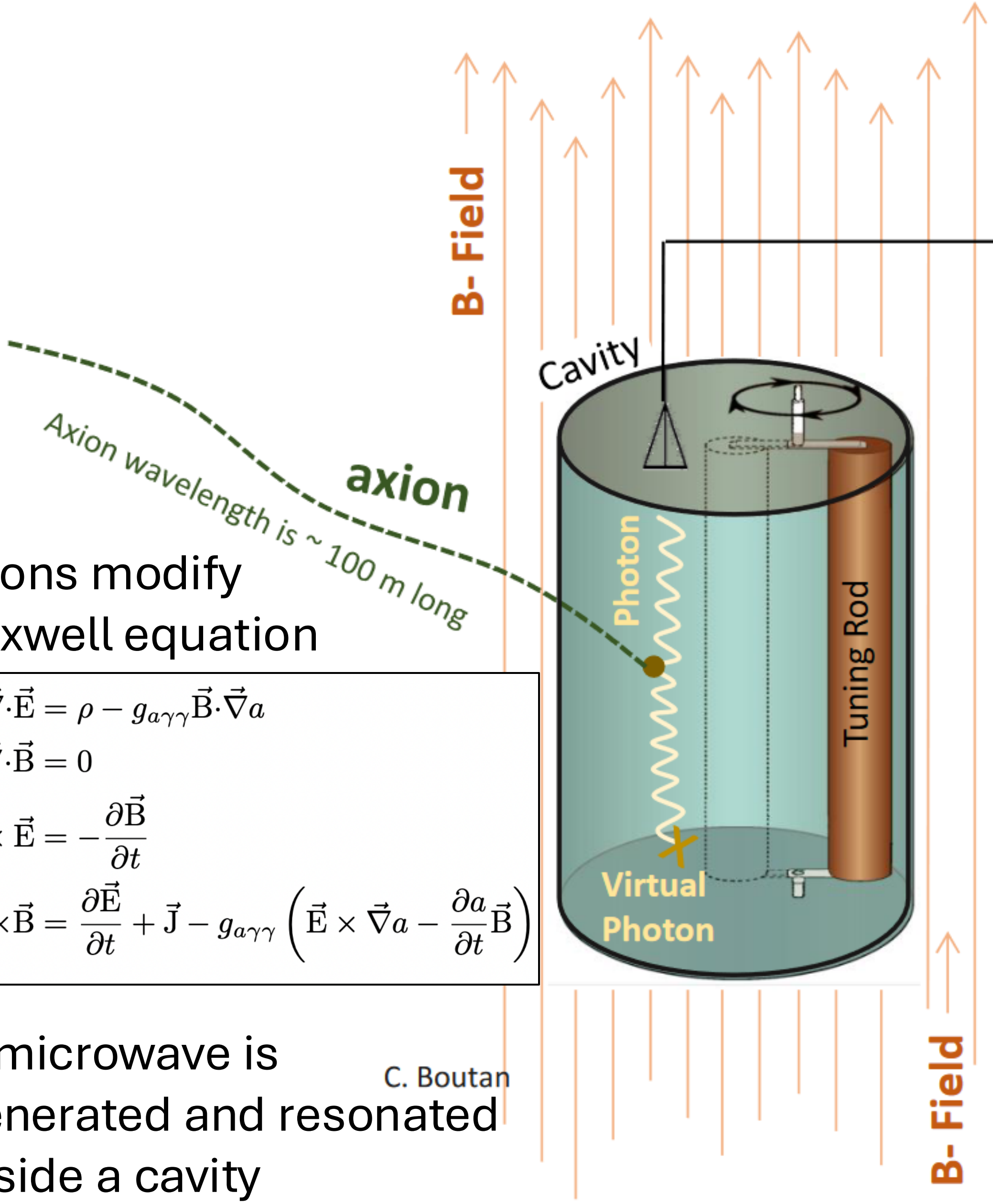

Microwaves converted from DM axion is very narrow band classical waves

# *Classical* electrodynamics is the mean to hunt axions

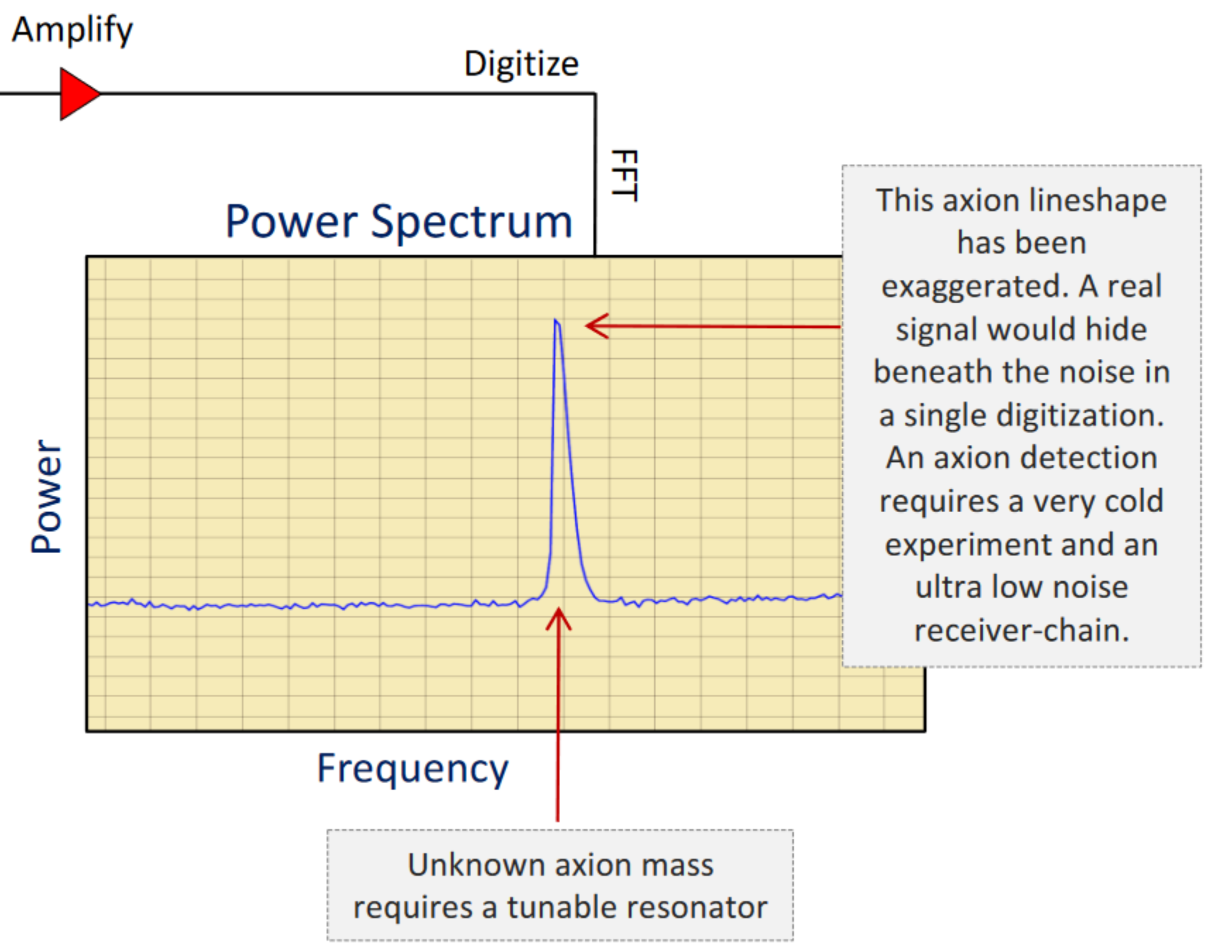
## *Skivie haloscope*

Axions modify Maxwell equation

$$\vec{\nabla}\cdot\vec{E} = \rho - g_{a\gamma\gamma}\vec{B}\cdot\vec{\nabla}a$$
$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\times\vec{B} = \frac{\partial\vec{E}}{\partial t} + \vec{J} - g_{a\gamma\gamma}\left(\vec{E}\times\vec{\nabla}a - \frac{\partial a}{\partial t}\vec{B}\right)$$

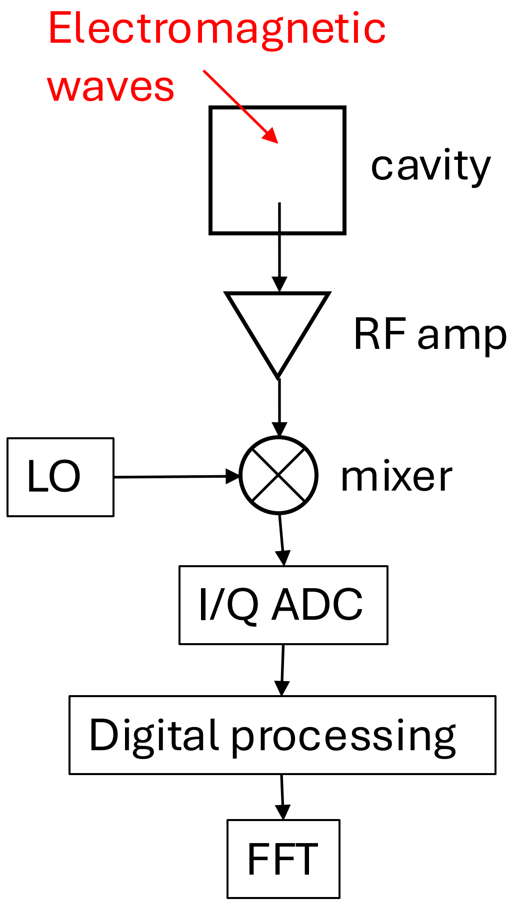


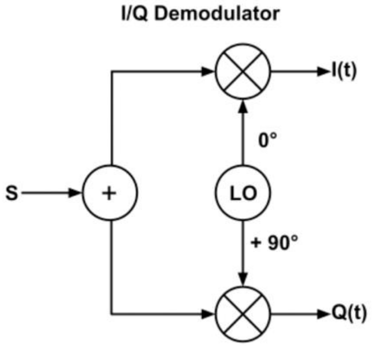

A microwave is generated and resonated inside a cavity



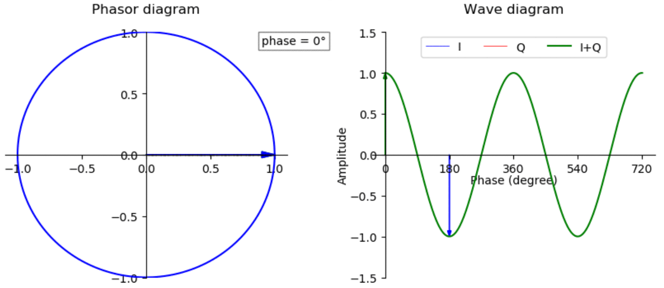

Courtesy: Gray Rybka, PATRAS2022

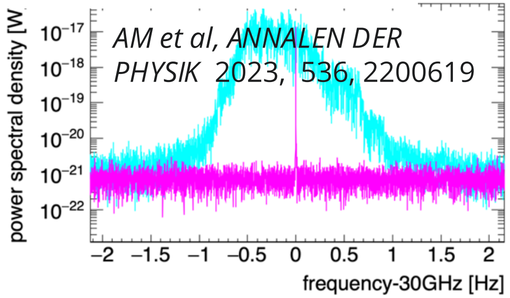
# Wave detection vs photon (energy) detection




From wikipedia

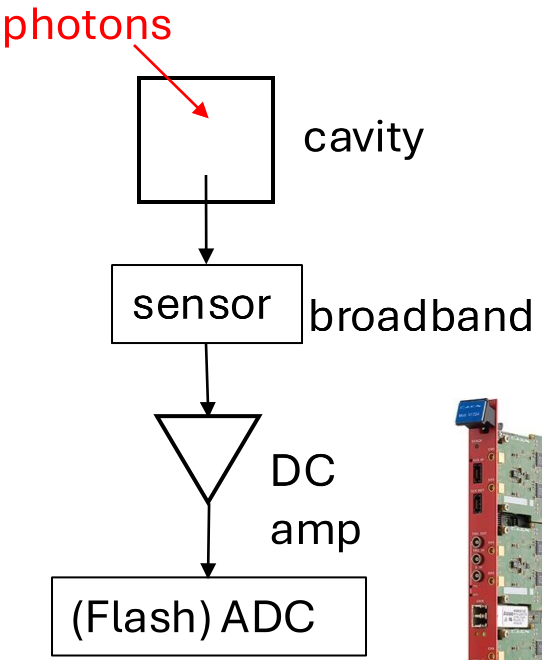


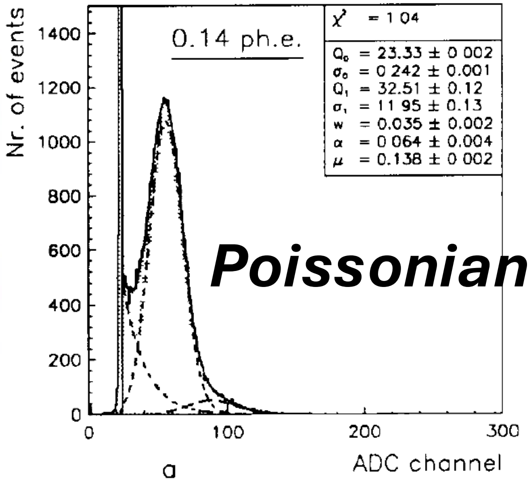

AM et al, ANNALEN DER PHYSIK 2023, 536, 2200619

$$RF(t) = I(t)\cos(\omega t) + Q(t)\sin(\omega t)$$

$$\rightarrow P(\omega) = \tilde{I}^2(\omega) + \tilde{Q}^2(\omega)$$








***Poissonian***

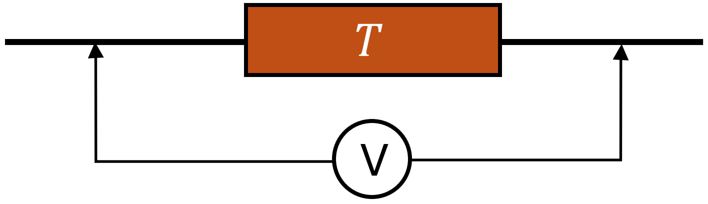
$$P(t) = n \times \hbar\omega \propto V_{ADC}\,(t)$$

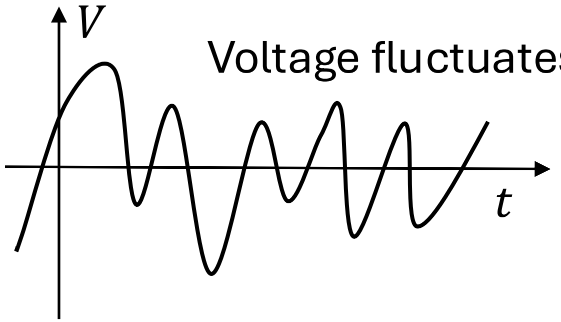
$$\Delta\phi\Delta n > 1$$

# Jonsson Nyquist Noise

J. B. Johnson Phys Rev 32 97 (1928): Experimental discovery of the relation
H. Nyquist Phys Rev 32 110 (1928): Thermodynamics + statistical mechanics of **bosonic** modes

Any conductor at temperature $T$




$$P_N = \frac{\langle V^2 \rangle}{4\mathrm{R}}$$

$$\langle V^2 \rangle \Delta\nu \sim 4R\Delta\nu \frac{h\nu}{e^{h\nu/k_B T} - 1} \xrightarrow[\text{Rayleigh Jeans}]{\mathrm{h}\nu \ll k_B T} 4Rk_B T\Delta\nu$$

Noise power spectral density $P_N = \frac{\langle V^2 \rangle}{4\mathrm{R}\Delta\nu} \sim k_B T$

→ "Blackbody radiation" of electromagnetic waves inside a 1D conductor

Quantum mechanical derivation H. B. Callen and T. A. Welton Phys Rev 1 34 (1951)

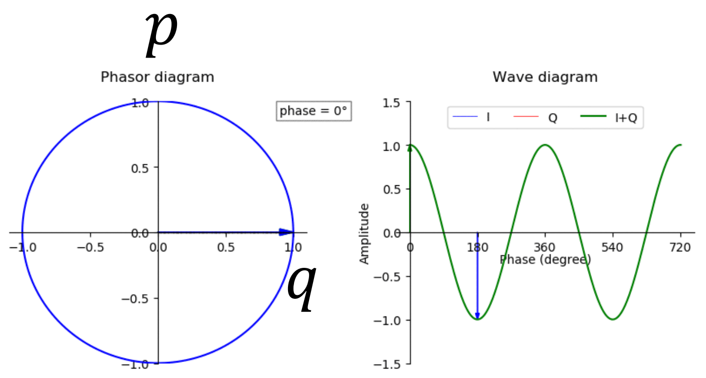
$$P_N = \left( \boxed{\frac{h\nu}{2}} + \frac{h\nu}{e^{h\nu/k_B T} - 1} \right) \quad [\mathrm{W/Hz}]$$

Zero-point energy

# Standard Quantum Limit from the Kennard inequality

$$[p,q] = \frac{i\hbar}{2} \rightarrow \langle \Delta p^2 \rangle \langle \Delta q^2 \rangle \geq |\langle [p,q] \rangle|^2 = \frac{\hbar}{4} \quad \text{(Kennard)}$$

$$[p_i, q_i] = \frac{i\hbar}{2} = [p_f, q_f]$$ : before and after the amplifier chain




$$q_f = Gq_i + q_g \qquad p_f = Gp_i + p_g$$

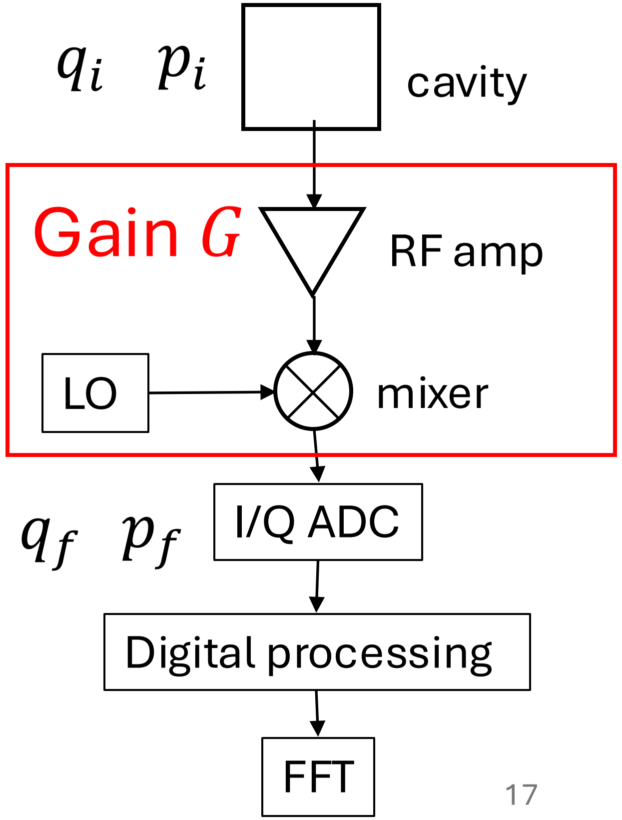
$$[p_f, q_f] = [{\color{red}G}p_0, {\color{red}G}q_0] + [p_g, q_g] = \frac{iG^2\hbar}{2} + [p_g, q_g] = \frac{i\hbar}{2}$$

$$\rightarrow [p_g, q_g] = \frac{i(1-G^2)\hbar}{2} \rightarrow \langle \Delta p_g^2 \rangle \langle \Delta q_g^2 \rangle \geq \frac{(G^2 - 1)\hbar^2}{4}$$

Amplifier uncertainty principle

$$P \geq \frac{1}{G^2}\left[\frac{G^2 h\nu}{2} + \frac{(G^2-1)h\nu}{2}\right] \xrightarrow{G \gg 1} 2 \times \frac{h\nu}{2} = h\nu$$

$P_{SQL} = h\nu$ : standard quantum limit




Ex) $h \times 1\ \mathrm{GHz} = 6.6 \times 10^{-25}$ W/Hz


S.K. Lamoreaux et al Phys Rev D 98 035020 (2013)
C. M. Caves PRD 26 8 1817 1982

# Photon detectors to overcome SQL

## Noise power of wave detection

$$P_l = h\nu(\bar{n} + \underset{\neq 1/2}{\mathbf{1}}\,)\sqrt{\frac{\Delta\nu}{t}} \propto \nu$$

## Noise power of photon detection

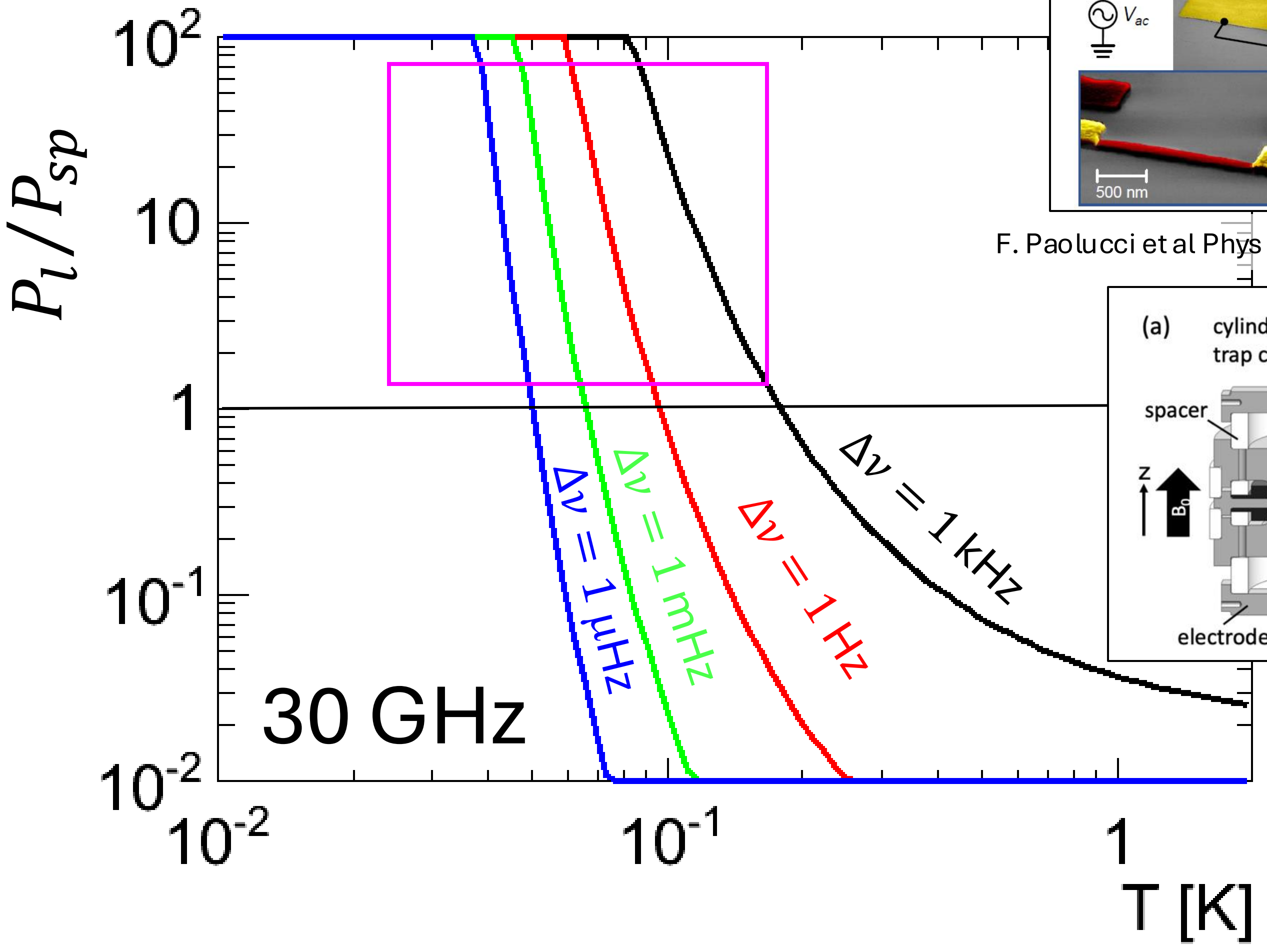
$$P_{sp} = h\nu\sqrt{\frac{\eta\bar{n}Q_c}{2\pi\nu t}} \propto \sqrt{\nu}$$

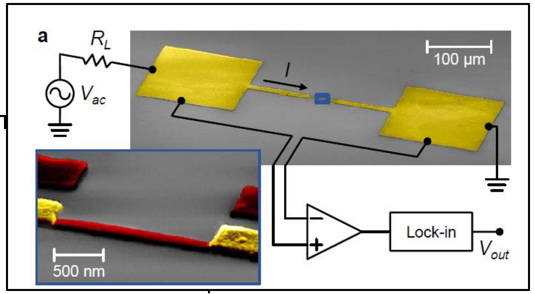
$$\bar{n} = \frac{1}{e^{h\nu/k_B T} - 1}$$

$Q_c$: cavity quality factor

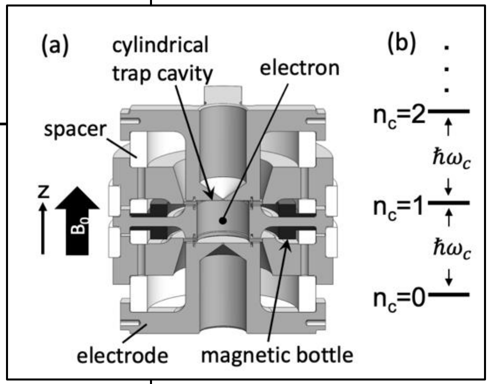
$\eta$: quantum efficiency of the sensor

S.K. Lamoreaux et al Phys Rev D 98 035020 (2013)




F. Paolucci et al Phys Rev Appl. 14 034055 (2020




X. Fan et al, PRL129, 261801, 2022

Single photon sensors may be a solution in the future

→ Although one loses phase information, zero background at cold may be better

→ Lower noise in **higher frequency** → where is the cross-over? 10 GHz? 100 GHz??

# Summary: pros and cons in wave vs particle

- Wave detection
  - Pros
    - Enormous enhancement of S/N by using narrow band nature of axion signal
      - Signal: narrow band of 20 kHz out of 20 GHz center frequency
      - Noise: broad band
    - Simple and established through commercial devices
    - OK at warm or 4 K level (in other words no use to be at mK)
  - Cons
    - Standard Quantum Limit (→ Vacuum squeezing to mitigate it…)
- Quantum / particle / energy detection
  - Pros
    - Free from SQL
  - Cons
    - No phase information → FFT not possible for narrow-band / broad band signal selection
    - Narrow band-pass filter / antenna is limited (resonator Q is the best)
    - Need cooling down way lower than wave detection (< 100 mK)

# Outline

- Background: classical to quantum detection
- Motivation of this talk: classical to quantum??
- Rigorous proof of classical to quantum
- Applications: coherence & homodyne with polarization
- Conclusion

# Question: classical → quantum

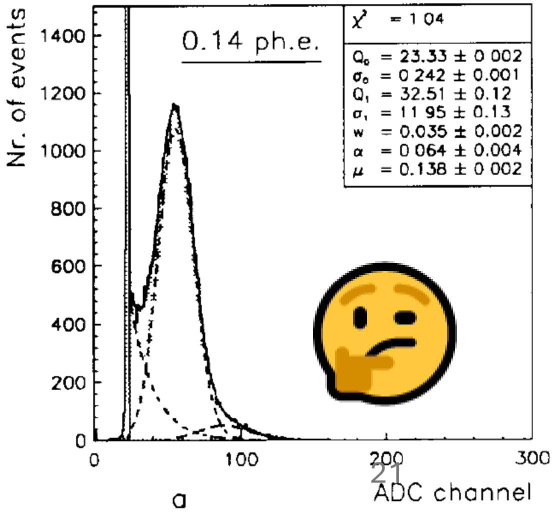
| | | |
|---|---|---|
| **Classical** axion | → **Classical** microwaves | → **Quantum** detection |

$$\epsilon \boldsymbol{\nabla} \cdot \mathbf{E} = \rho - g_{a\gamma} \mathbf{B}_{\mathrm{e}} \cdot \boldsymbol{\nabla} a \,,$$

$$\boldsymbol{\nabla} \times \mathbf{H} - \dot{\mathbf{E}} = \mathbf{J} + g_{a\gamma} \mathbf{B}_{\mathrm{e}} \dot{a} \,,$$

$$\ddot{a} - \boldsymbol{\nabla}^2 a + m_a^2 a = g_{a\gamma} \mathbf{E} \cdot \mathbf{B}_{\mathrm{e}} \,,$$

What is the quantum statistical distribution of detected photons?
→ Goal: Monte Carlo




- Can we obtain quantum statistical distribution without assuming quantum nature in the source?
- Can we assume a Poisson distribution?
- **What kind of quantum state gives Poisson distribution**

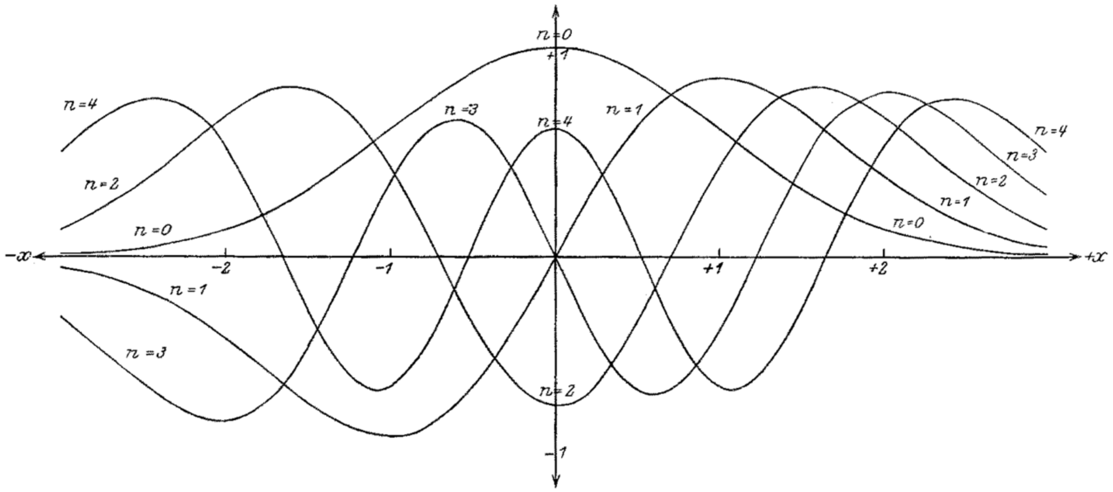
# Der stetige Übergang von der Mikro- zur Makromechanik.

Von E. SCHRÖDINGER, Zürich.

$$\frac{m}{2}\left(\frac{dq}{dt}\right)^2 + 2\pi^2\nu_0^2 m q^2$$

$$\left.\begin{array}{l}\psi_n = e^{-\frac{x^2}{2}} H_n(x)\, e^{2\pi i \nu_n t}\\ \left(\nu_n = \dfrac{2n+1}{2}\nu_0;\ n = 0,\ 1,\ 2,\ 3 \dots\right)\end{array}\right\} \quad (3)$$

Die $H_n$ sind die nach HERMITE benannten

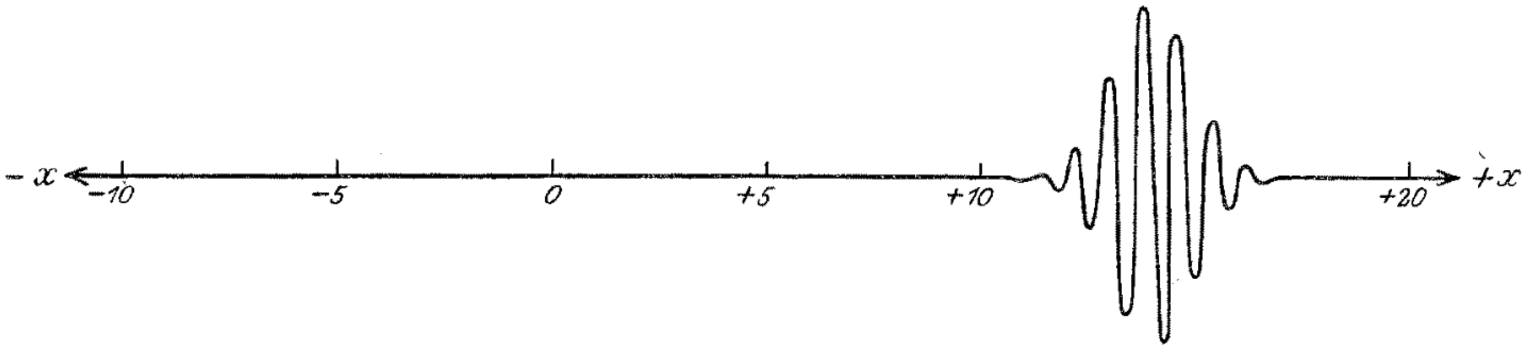



$$\psi = \sum_{n=0}^{\infty}\left(\frac{A}{2}\right)^n \frac{\psi_n}{n!}$$

$$= e^{\pi i \nu_0 t}\sum_{n=0}^{\infty}\left(\frac{A}{2}\, e^{2\pi i \nu_0 t}\right)^n \frac{1}{n!}\, e^{-\frac{x^2}{2}} H_n(x)\,.$$

$$\psi = e^{\boxed{\frac{A^2}{4} - \frac{1}{2}(x - A\cos 2\pi\nu_0 t)^2}}$$

$$\cdot \cos\left[\pi\nu_0 t + (A\sin 2\pi\nu_0 t)\cdot\left(x - \frac{A}{2}\cos 2\pi\nu_0 t\right)\right]$$




Schrödinger, E. (1926). "Der stetige Übergang von der Mikro- zur Makromechanik". Die Naturwissenschaften 14 (28)

# Very classical quantum state: Schrödinger's wave packet

$$\Psi(x_1) = \frac{1}{\sqrt{2\pi}} e^{-(x_1 - \alpha \cos \omega t)^2}$$

$$\alpha \Psi(x_1) = \left[ x_1 + \frac{1}{2} \frac{d}{d x_1} \right] \Psi(x_1)$$

$\alpha$: eigenvalue of annihilation operator

$$\hat{a} \equiv \hat{x}_1 + i \hat{x}_2$$

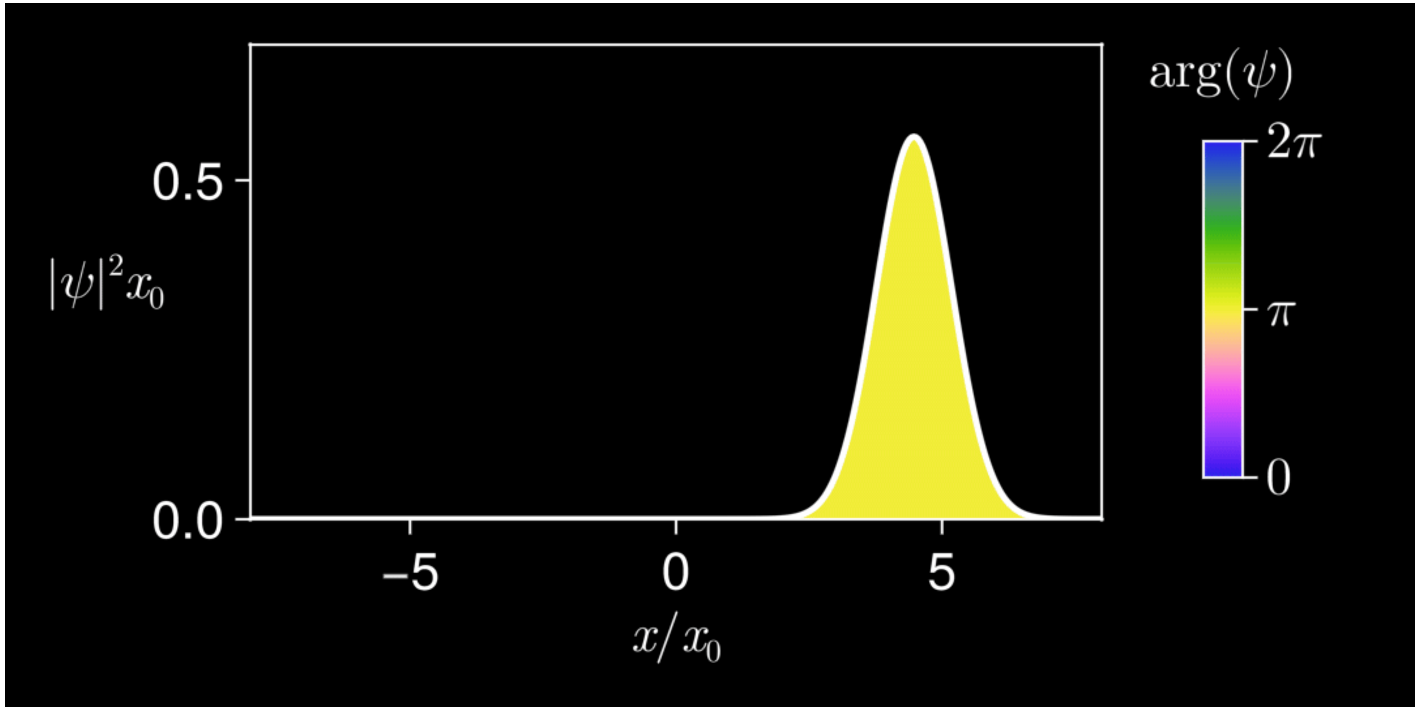
$$a|\alpha\rangle = \alpha|\alpha\rangle$$

<ins>Hints for derivation</ins>

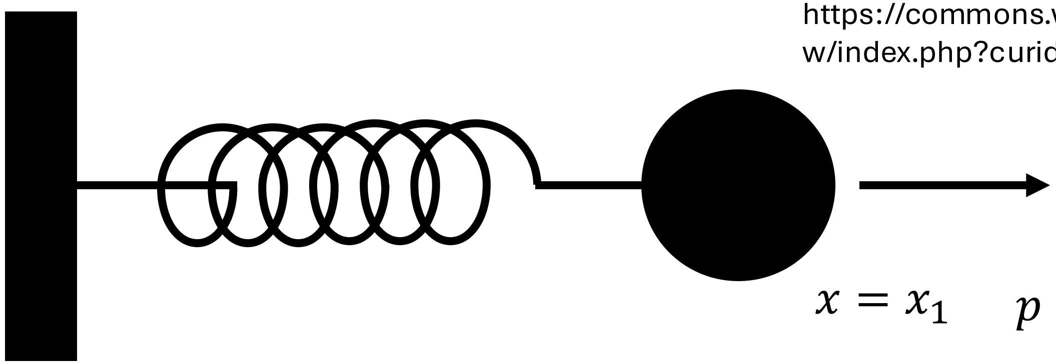
$\Psi(x_1) \equiv \langle x_1 | \alpha \rangle$

$\langle x_1 | \hat{x}_1 | x_1 \rangle = x_1$

$[\hat{x}_1, \hat{x}_2] = \frac{i}{2} \rightarrow \langle x_1 | \hat{x}_2 | x_1 \rangle \equiv \frac{1}{2i} \frac{d}{d x_1}$



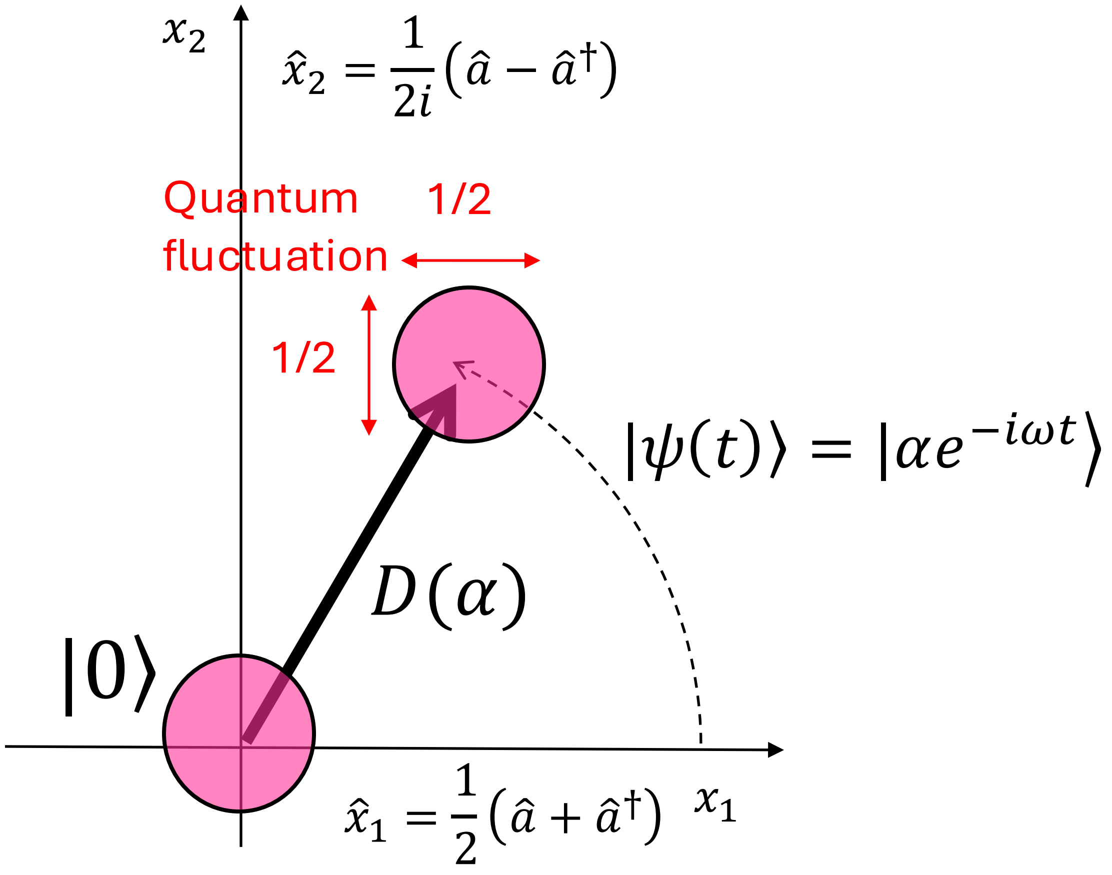


$x = x_1 \qquad p = \frac{\partial H}{\partial x} = x_2$

# Modern representation of wave packet: coherent state

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$




## Displacement operator

$$D(\alpha) \equiv \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$$

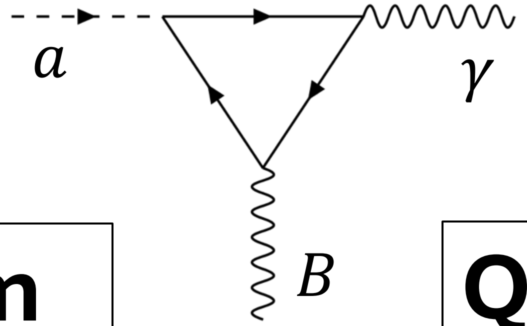
$$|\alpha\rangle = D(\alpha)|0\rangle$$

## Electric field operator

$$\hat{\mathcal{E}}(\boldsymbol{r}, t) = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left[\hat{a} e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} - \hat{a}^\dagger e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}\right]$$

$$\begin{cases} E \equiv \langle\alpha|\hat{\mathcal{E}}(\boldsymbol{r}, t)|\alpha\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left[\alpha e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} - \alpha^* e^{+i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}\right] & \text{: Plane wave solution of Maxwell equation} \\ P(n) = |\langle n|\alpha\rangle|^2 = \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2} & \text{: Poisson distribution} \end{cases}$$

# Don’t we need a quantum state of axion?




**Quantum** axion → **Quantum** microwaves → **Quantum** detection

$$\hat{H} = \omega_a \hat{a}_a^\dagger \hat{a}_a + \omega_\gamma \hat{a}_\gamma^\dagger \hat{a}_\gamma + \omega_m \hat{a}_a^\dagger \hat{a}_\gamma + \omega_m^* \hat{a}_\gamma^\dagger \hat{a}_a$$

PHYSICAL REVIEW D VOLUME 45, NUMBER 5 1 MARCH 1992

**Quantum statistics in particle mixing phenomena**

G. Raffelt, G. Sigl, and L. Stodolsky
*Max-Planck-Institut für Physik, Postfach 401212, W-8000 München 40, Germany*

PREPARED FOR SUBMISSION TO JCAP

CERN-TH-2017-078
MPP-2017-85

**Axion-photon conversion caused by dielectric interfaces: quantum field calculation**

Ara N. Ioannisian,[a,b] Narine Kazarian,[b] Alexander J. Millar[c] and Georg G. Raffelt[c]

- Mean is OK but standard deviation depends on the assumptions on the source
- Simplest example: coherent state → coherent state: $|\alpha_a, \alpha_\gamma\rangle = (1 + U_{a\gamma}\alpha_a \alpha_\gamma^\dagger)|\alpha_a, 0\rangle$

# Example of quantum nature of axions 1: squeezing

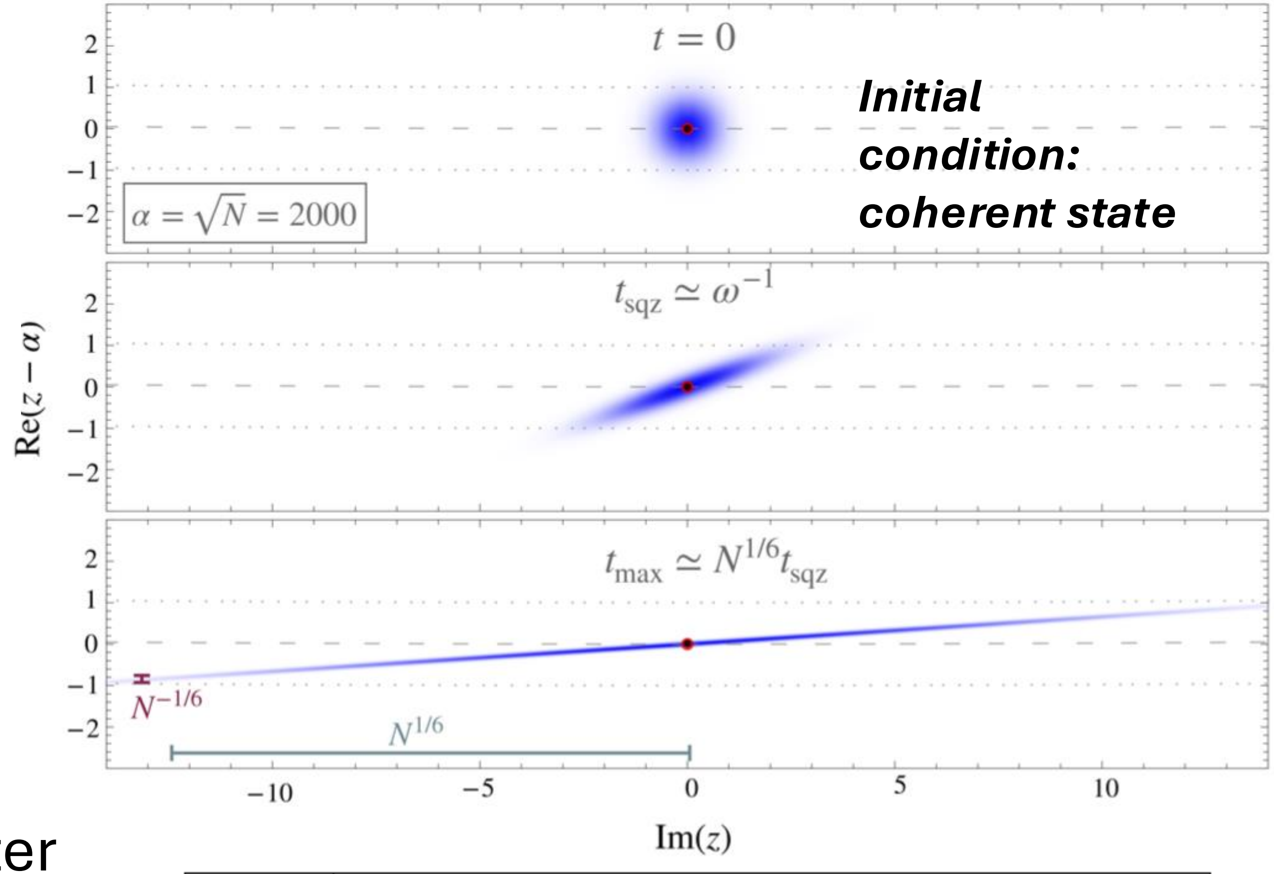
$$i\hbar\partial_t\hat{\psi}(\boldsymbol{x},t) = -\frac{\hbar^2}{2mA(t)^2}\nabla^2\hat{\psi}(\boldsymbol{x},t) + m\boxed{\hat{\Phi}(\boldsymbol{x},t)}\hat{\psi}(\boldsymbol{x},t)$$

$$\boxed{\nabla^2\hat{\Phi}(\boldsymbol{x},t)} = \frac{4\pi G m}{A(t)}\left(\hat{\psi}^\dagger(\boldsymbol{x},t)\hat{\psi}(\boldsymbol{x},t) - \overline{\hat{\psi}^\dagger(\boldsymbol{x},t)\hat{\psi}(\boldsymbol{x},t)}\right), \tag{1}$$

Gravitational potential

$$i\partial_t\hat{a}(t) = \omega(t)\hat{a}(t) + \boxed{2\chi(t)\hat{a}^\dagger(t)\hat{a}(t)\hat{a}(t)}$$

Kerr-type nonlinearity




| | Cosmology | Solitonic core | Milkyway |
|---|---|---|---|
| $t_{\rm sqz}\,[\mu s]$ | $66\left(\frac{10^{-5}\mathrm{eV}}{m}\right)$ | $1400\left(\frac{10^{-5}\mathrm{eV}}{m}\right)$ | $33\left(\frac{10^{-5}\mathrm{eV}}{m}\right)$ |
| $t_{\rm max}\,[\mathrm{yr}]$ | $3500\left(\frac{10^{-5}\mathrm{eV}}{m}\right)^{\frac{7}{6}}$ | $0.5\left(\frac{10^{-5}\mathrm{eV}}{m}\right)^{\frac{4}{3}}$ | $33000\left(\frac{10^{-5}\mathrm{eV}}{m}\right)^{\frac{7}{6}}$ |
| $r_{\rm max}$ | $36+\frac{1}{6}\ln\left(\frac{10^{-5}\mathrm{eV}}{m}\right)$ | $24+\frac{1}{3}\ln\left(\frac{10^{-5}\mathrm{eV}}{m}\right)$ | $32+\frac{1}{6}\ln\left(\frac{10^{-5}\mathrm{eV}}{m}\right)$ |
| $t_{\rm Ehr}\,[\mathrm{yr}]$ | $10^{35}\left(\frac{10^{-5}\mathrm{eV}}{m}\right)^{\frac{3}{2}}$ | $10^{21}\left(\frac{10^{-5}\mathrm{eV}}{m}\right)^{2}$ | $10^{32}\left(\frac{10^{-5}\mathrm{eV}}{m}\right)^{\frac{3}{2}}$ |

- Gravitational self-interaction of dark matter axions causes quantum squeezing within a surprisingly short period
- Experimental detection seems non-feasible ☹

Phys. Rev. D 106, 043517, 2022

# Example quantum nature of axions 2 : thermalization

$$\mathcal{L}[a,\chi] = \frac{1}{2}\partial_\mu a(x)\partial^\mu a(x) - \frac{1}{2}m_a^2\, a^2(x) - \boxed{g a(x)\, \mathcal{O}_\chi(x) + \mathcal{L}_\chi}$$

Interaction to environment (SM)

$$a(\vec{x}, t=0) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}}\left[b_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}} + b^\dagger_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}}\right]$$

$$n(\omega_q) = \frac{1}{e^{\beta\,\omega_q} - 1}$$

$$\hat{\rho}(t) = e^{-iHt}\hat{\rho}(0)e^{iHt}$$ Time evolution of density matrix

$$\rho(0) = \rho_a(0)\otimes\rho_\chi(0)$$ Initial condition: axion x environment

$$\rho_\chi(0) = \frac{e^{-\beta H_\chi}}{\mathrm{Tr}e^{-\beta H_\chi}}$$ Environment: thermal bath (CMB)

$$\rho_a(0) = |\Delta\rangle\langle\Delta|$$ Initial axion: coherent state from misalignment mechanism

$$|\Delta\rangle = \Pi_{\vec{k}}\, e^{-\frac{1}{2}|\Delta_{\vec{k}}|^2}\, e^{-\Delta_{\vec{k}} b^\dagger_{\vec{k}}}\,|0\rangle$$

<u>Quantum master equation</u>

→ Initial coherent axions become decoherent due to thermalization

$$\mathcal{E}(t) = \frac{1}{V}\sum_{\vec{q}} N_q(t)\omega_q = \frac{1}{2}\left[\overline{\pi}^2 + m_a^2\,\overline{a}^2\right]\boxed{e^{-\gamma_0 t}} + \int\frac{d^3q}{(2\pi)^3}\,\omega_q\, n(\omega_q)\boxed{(1-e^{-\gamma_q t})}$$

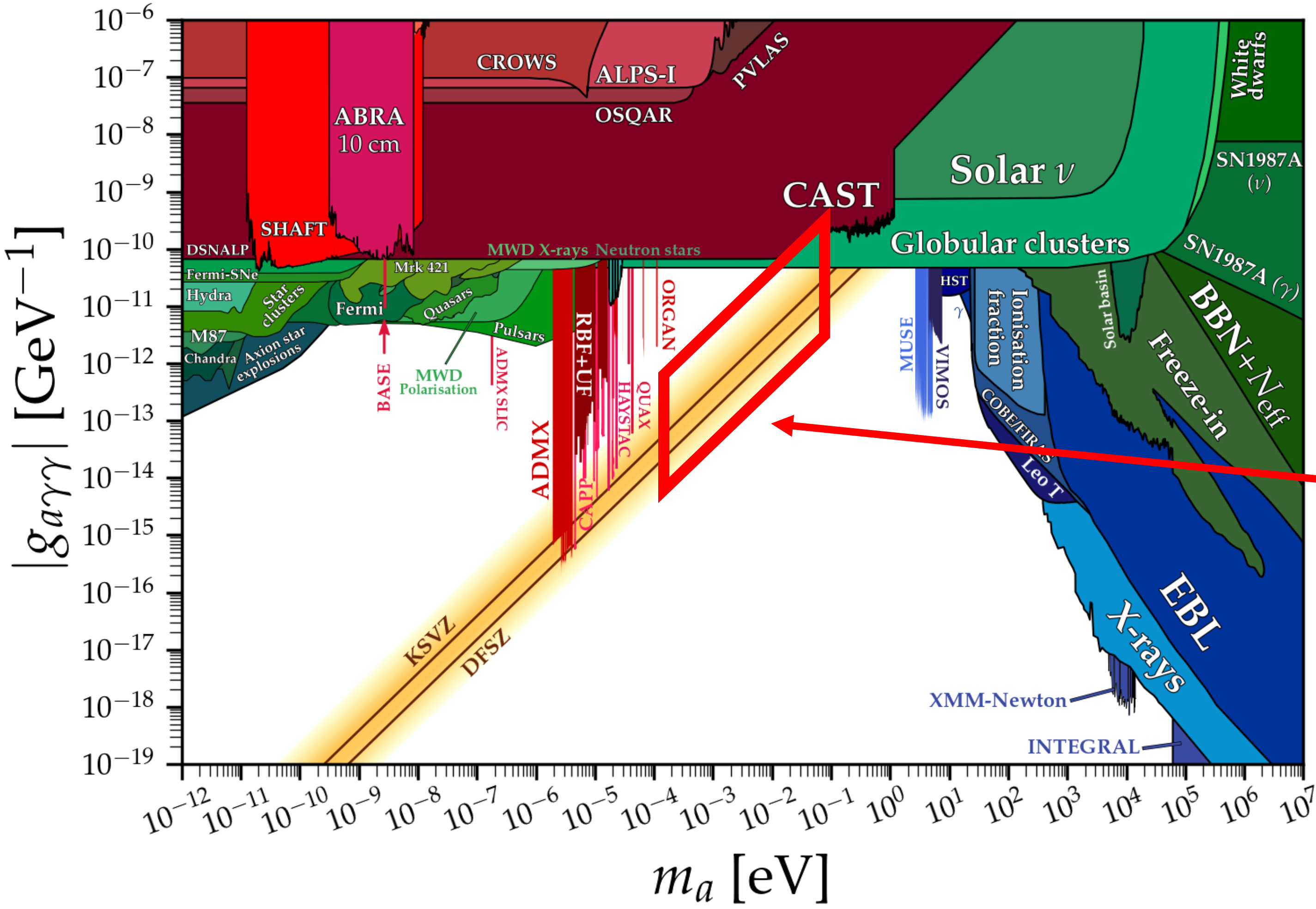
Coherent term decays

Thermal term grows

Decay rate: $$\gamma_q = \Gamma_q(\infty) = \frac{g^2}{2\omega_q}\rho(\omega_q, q)$$

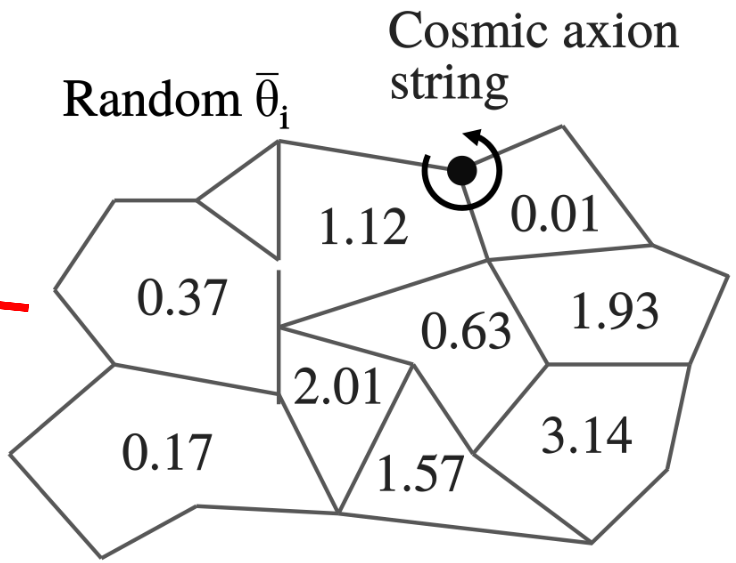
Phys. Rev. D 107, 063518 (2023)

# Example quantum nature of axions 3: decay products

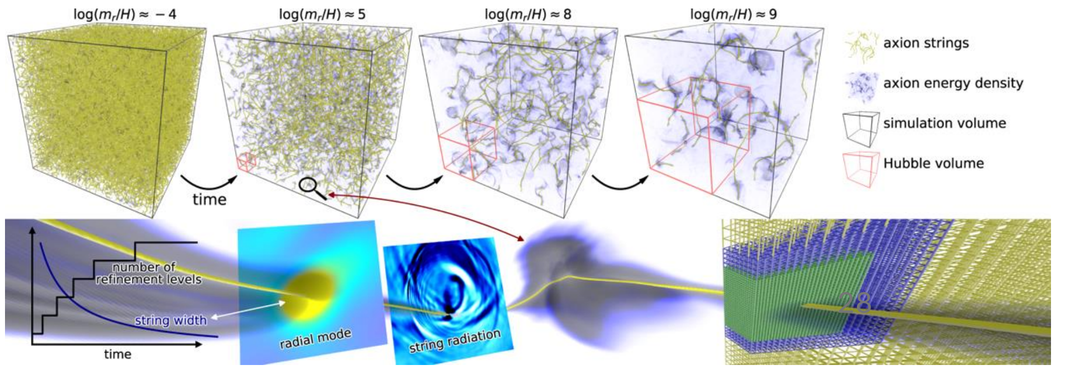



## PQ scale < inflation

Our Universe:




S. Knirck PhD thesis

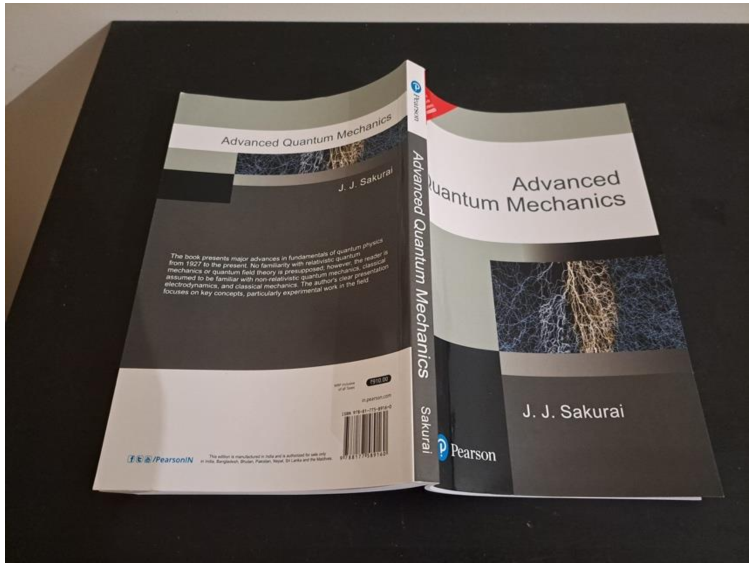



Axion DM is made of decay products of axion strings

Buschmann, *et al.* *Nat Commun* **13**, 1049 (2022)

# 杞憂っぽい

$$E_{cl}^2 = \frac{\bar{n}}{\lambda} \gg \frac{1}{\lambda^4} \sim \langle 0 | \hat{E} \cdot \hat{E} | 0 \rangle$$




任意の古典的アクシオンから光子の量子状態を予言できる

# Outline

- Background: classical to quantum detection
- Motivation of this talk: classical to quantum??
- Rigorous proof of classical to quantum
- Applications: coherence & homodyne with polarization
- Conclusion

# Glauber's theorem: classical → quantum

PHYSICAL REVIEW VOLUME 131, NUMBER 6 15 SEPTEMBER 1963

**Coherent and Incoherent States of the Radiation Field***

ROY J. GLAUBER
*Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts*
(Received 29 April 1963)

$$H_1(t)=-\frac{1}{c}\int \mathbf{j}(\mathbf{r},t)\cdot\mathbf{A}(\mathbf{r},t)d\mathbf{r}. \quad (9.16)$$

The introduction of an explicitly time-dependent interaction of this type means that the state vector for the field, $|\ \rangle$, which previously was fixed (corresponding to the Heisenberg picture) will begin to change with time in accordance with the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\ \rangle=H_1(t)|\ \rangle, \quad (9.17)$$

which is the one appropriate to the interaction representation. The solution of this equation is easily found.[20] If we assume that the initial state of the field at time $t=-\infty$ is one empty of all photons, then the state of the field at time $t$ may be written in the form

$$|t\rangle=\exp\left\{\frac{i}{\hbar c}\int_{-\infty}^{t}dt'\int \mathbf{j}(\mathbf{r},t')\cdot\mathbf{A}(\mathbf{r},t')d\mathbf{r}+i\varphi(t)\right\}|\mathrm{vac}\rangle. \quad (9.18)$$

$$D_k(\beta_k)=\exp[\beta_k a_k^\dagger-\beta_k^* a_k]. \quad (9.19)$$

Then it is clear from the expansion (2.10) for the vector potential that we may write

$$\exp\left\{\frac{i}{\hbar c}\int_{-\infty}^{t}dt'\int \mathbf{j}(\mathbf{r},t')\cdot\mathbf{A}(\mathbf{r},t')d\mathbf{r}\right\}=\prod_k D_k[\alpha_k(t)], \quad (9.20)$$

where the time-dependent amplitudes $\alpha_k(t)$ are given by

$$\alpha_k(t)=\frac{i}{(2\hbar\omega)^{1/2}}\int_{-\infty}^{t}dt'\int d\mathbf{r}\,\mathbf{u}_k^*(\mathbf{r})\cdot\mathbf{j}(\mathbf{r},t')e^{i\omega t'}. \quad (9.21)$$

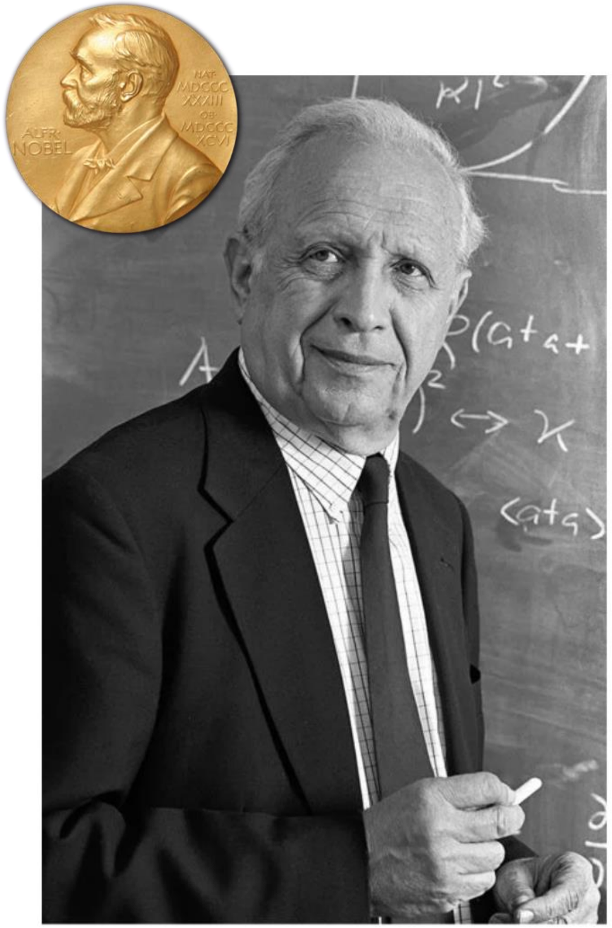
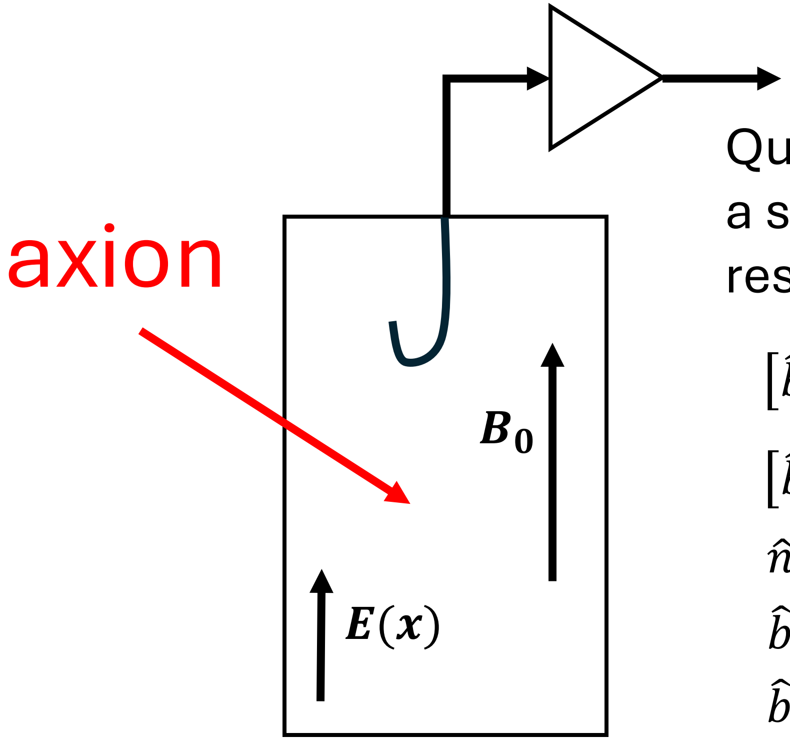


Photo: J.Reed

Classical current generates quantized photons in a quantum coherent state
→ **Let's apply this to classical axions**

# **Any** classical axion generates photons in a quantum coherent state




Quantized photon states of a single mode inside a resonant cavity

$$[\hat{b}, \hat{b}^\dagger] = i\hbar$$

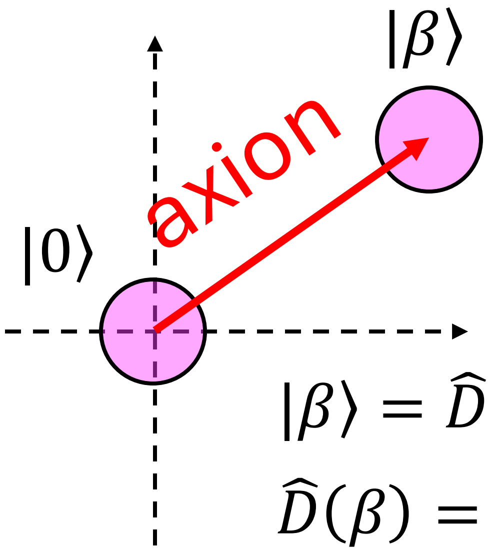
$$[\hat{b}, \hat{b}] = [\hat{b}^\dagger, \hat{b}^\dagger] = 0$$

$$\hat{n}|n\rangle \equiv \hat{b}^\dagger \hat{b}|n\rangle = n|n\rangle$$

$$\hat{b}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$\hat{b}|n\rangle = \sqrt{n}|n-1\rangle$$

Quantum state of photons from axions




Classical axion ~ displacement operator (x phase)

$$|\beta\rangle = \hat{D}(\beta)|0\rangle$$

$$\hat{D}(\beta) = \exp(\beta\hat{b}^\dagger - \beta^*\hat{b})$$

$$\hat{H} = \underbrace{\boxed{\hbar\omega\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right)}}_{\hat{H}_0} + \underbrace{\boxed{\hbar[f(t)\hat{b}^\dagger + f^*(t)\hat{b}]}}_{\hat{V}_I}$$

Only linear operators

$$f(t) = g_{a\gamma\gamma}\boxed{a(t)}\int (\boldsymbol{E}(x)\cdot\boldsymbol{B}_0)d^3x$$

Any classical axion field

Exact solution without perturbation

$$\begin{cases} |\psi(t)\rangle = \exp(-iC_1(t))\boxed{\hat{D}(\beta(t))}\exp(-iH_0 t/\hbar)\,|\psi(0)\rangle \\ \beta(t) = C_2(t)\exp(-i\omega t) \\ C_1(t) = -\dfrac{i}{2}\displaystyle\int_0^t dt' f^*(t')\exp(i\omega t')\int_0^{t'} dt'' f(t'')\exp(i\omega t'') \\ C_2(t) = \displaystyle\int_0^t dt' f(t')\exp(i\omega t') \end{cases}$$

**Derived by Ayuki Kamada U Warsaw**

# Key takeaways

- Interaction term is linear in $\hat{b}^\dagger$ and $\hat{b}$
  - Rigorous solution is given by a displacement operator and a phase shift
- The displacement operator generates a coherent state from vacuum
  - With complex amplitude $\beta(t)$ given by time integral of the classical axion field $f(t)$
- Photon statistic in a coherent state is given by a Poisson distribution
  - With mean number of photons $\bar{n} = |\beta(t)|^2$
- Valid for any classical axions and any initial state of photons not only the vacuum state

# Initially in the vacuum state → coherent state

Definition of the label of the coherent state

$$\boxed{\beta(t) = \exp(-i\omega t)\int_0^t dt' f(t')\exp(i\omega t')}$$

$$|\psi(t)\rangle = \exp\big(-iC_1(t)\big)\hat{D}\big(\beta(t)\big)\,|0\rangle = \exp\big(-iC_1(t)\big)\,|\beta(t)\rangle$$

<u>Photon counting (Projective measurement of photon numbers)</u>

Property of a coherent state

$$|\beta\rangle = \exp\left(-\frac{|\beta|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!}}|n\rangle$$

$$P_n = |\langle n|\psi(t)\rangle|^2 = |\langle\psi(t)|\hat{n}|\psi(t)\rangle|^2 = \langle\beta(t)|\hat{n}|\beta(t)\rangle = \frac{e^{-|\beta(t)|^2}|\beta(t)|^{2n}}{n!}$$

- The number of photons from classical axion conversion obeys **Poisson distribution of time varying mean**
- The time variation is purely from the time integral of classical axion field

<u>Field measurement (Projective measurement of field operator)</u>

Electric field operator

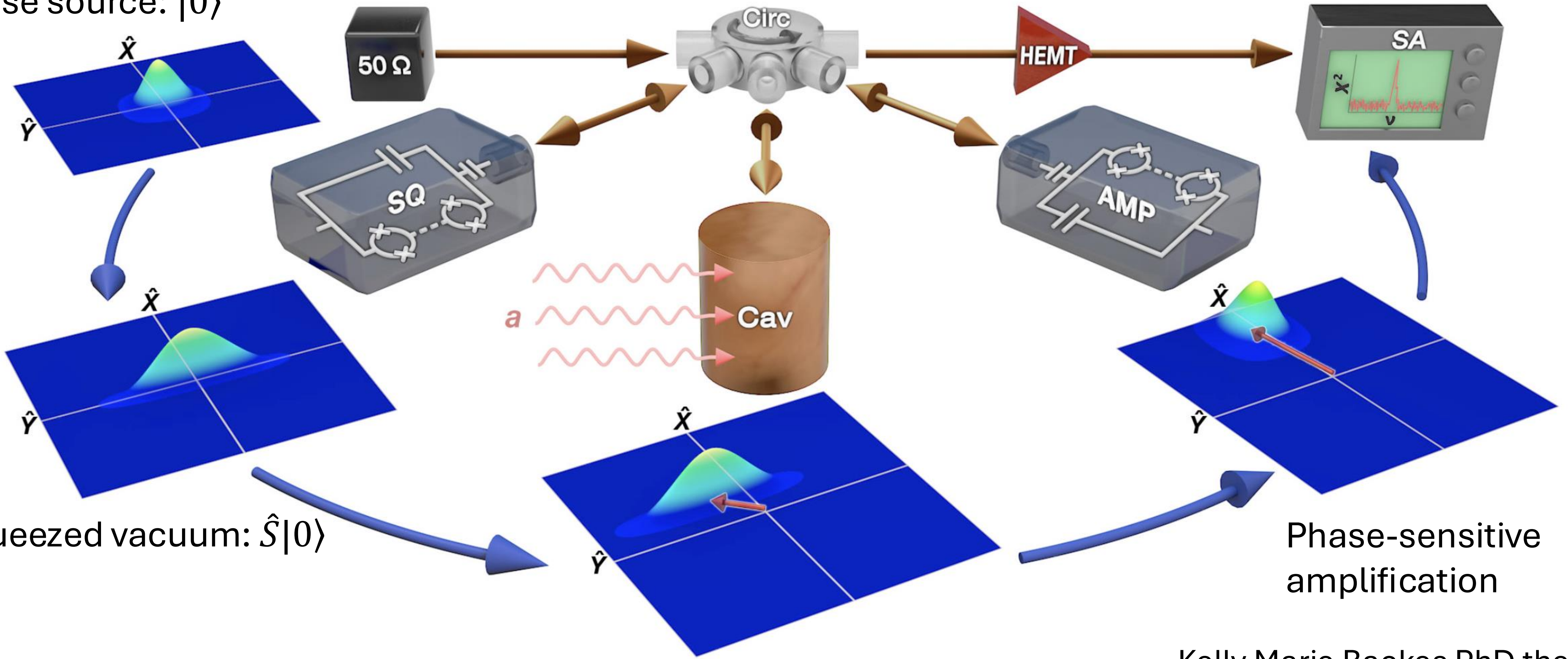
$$\hat{\mathcal{E}}(\boldsymbol{r},t) = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left[\hat{b}e^{-i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})} - \hat{b}^\dagger e^{i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})}\right]$$

$$E(t) \equiv \left\langle\psi(t)\middle|\hat{\mathcal{E}}(\boldsymbol{r},t)\middle|\psi(t)\right\rangle = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left[\beta(t)e^{-i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})} - \beta^*(t)e^{+i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})}\right]$$

- This is the classical electric field (solution of Maxwell equation) → Classical Primakoff effect is reproduced
- The line-width information is implemented inside $\beta(t)$ → **time varying amplitude in time domain**
- Integration time $t$ (FFT time) needs to be selected depending on the coherent time of the classical axion field $a(t)$

# Initially in the squeezed vacuum state

Noise source: $|0\rangle$




Squeezed vacuum: $\hat{S}|0\rangle$

Phase-sensitive amplification

Coherent displacement by a classical axion signal

Kelly Marie Backes PhD thesis
Nature, 590(7845):238–242

# Initially in a thermal state → coherent thermal state

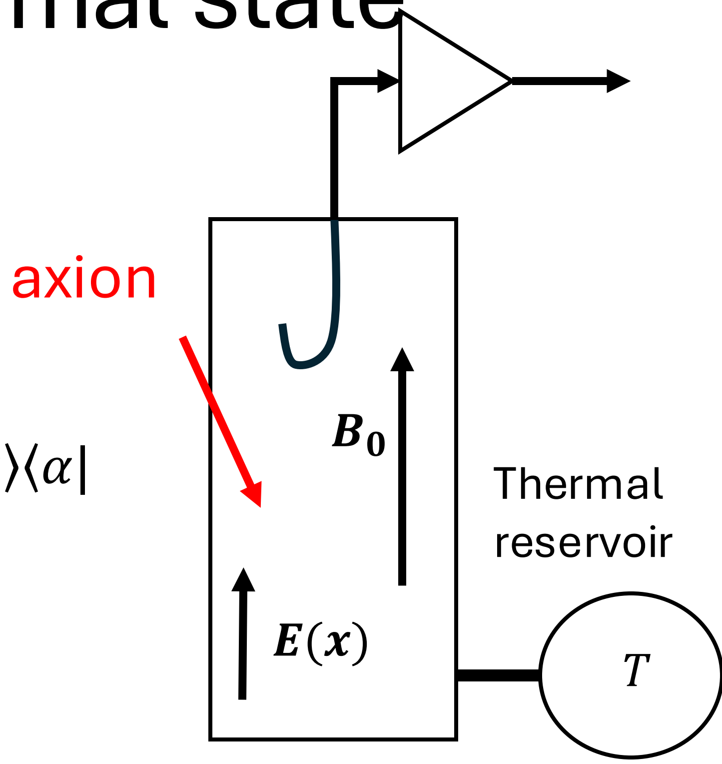
Unitary operator of time evolution

$$U(t) = \exp\left(-iC_1(t)\right)\hat{D}\left(\beta(t)\right)$$

$$\bar{n}_{th} = \frac{1}{\exp(\hbar\omega/k_B T) - 1}$$

$$\hat{\rho}_0 \equiv \frac{\exp\left(-\hat{H}_0/k_B T\right)}{Z} = \frac{1}{1+\bar{n}_{th}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}_{th}}{1+\bar{n}_{th}}\right)^n |n\rangle\langle n| = \int d^2\alpha \, \frac{\exp(-|\alpha|^2/\bar{n})}{\pi\bar{n}} |\alpha\rangle\langle\alpha|$$

$$Z = \mathrm{Tr}\left[\exp\left(-\hat{H}_0/k_B T\right)\right]$$




Coherent thermal state

Realistic quantum model of axion + noise

$$\hat{\rho}(t) = U(t)\hat{\rho}_0 U^\dagger(t) = \frac{1}{Z}\hat{D}\left(\beta(t)\right)\exp\left(-\hat{H}_0/k_B T\right)\hat{D}^\dagger\left(\beta(t)\right)$$

Statistics of photons

$$\langle A\rangle_{th} = \frac{1}{Z} Tr\{e^{-\hbar\omega b^\dagger b/k_B T} A\}$$

$$\bar{n} = |\beta(t)|^2 + \bar{n}_{th}$$ **= Poisson + Planck**

$$\sigma^2 = \sigma_{th}^2 + |\beta|^2\left(1 + \boxed{2\left\langle b^\dagger b\right\rangle_{th}}\right)$$ **≠ Poisson + Planck**

**→ Naïve additive Monte Carlo (Poissonian signal + thermal noise) is wrong!**

# Cf) Multi-mode Planck → reduction to Poissonian

M. Fox "Quantum Optics" p.85

$$\mathcal{P}_\omega(n) = \frac{1}{\overline{n}+1}\left(\frac{\overline{n}}{\overline{n}+1}\right)^n. \quad (5.29)$$

This distribution is called the **Bose–Einstein distribution**. From eqn 5.26 we can see that $\mathcal{P}_\omega(n)$ is always largest for $n = 0$, and decreases exponentially for increasing $n$.

Figure 5.5 compares the photon statistics for a single mode of thermal light with the Bose–Einstein distribution to those of a Poisson distribution with the same value of $\overline{n}$. It is apparent that the distribution of photon numbers for thermal light is much broader than for Poissonian light. This is hardly surprising, given the nature of thermal energy fluctuations. The variance of the Bose–Einstein distribution can be found by inserting $\mathcal{P}_\omega(n)$ from eqn 5.29 into eqn 5.14, giving (see Exercise 5.3):

$$(\Delta n)^2 = \overline{n} + \overline{n}^2. \quad (5.30)$$

This shows that the variance of the Bose–Einstein distribution is always larger than that of a Poisson distribution (cf. eqn 5.15), and that thermal light therefore falls into the category of super-Poissonian light defined

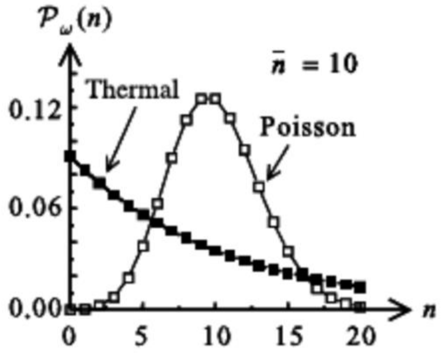



**Fig. 5.5** Comparison of the photon statistics for a single mode of a thermal source and a Poisson distribution with the same value of $\overline{n} = 10$.

STD of thermal noise

$$\sigma^2 = \bar{n} + \frac{\bar{n}^2}{N_m}$$

Poisson

Deviation from Poisson

$N_m$: number of modes
→ $N_m \gg 1$ with bandwidth

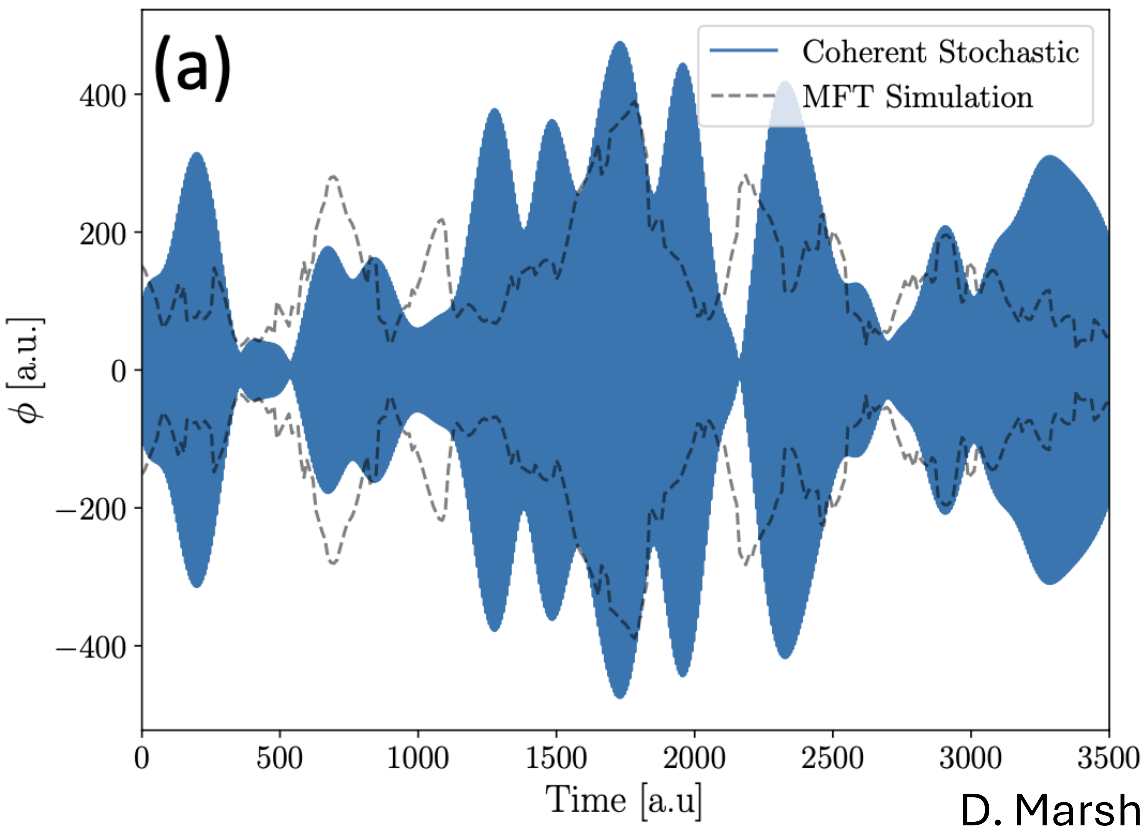
$$\sigma^2 \to \bar{n}$$

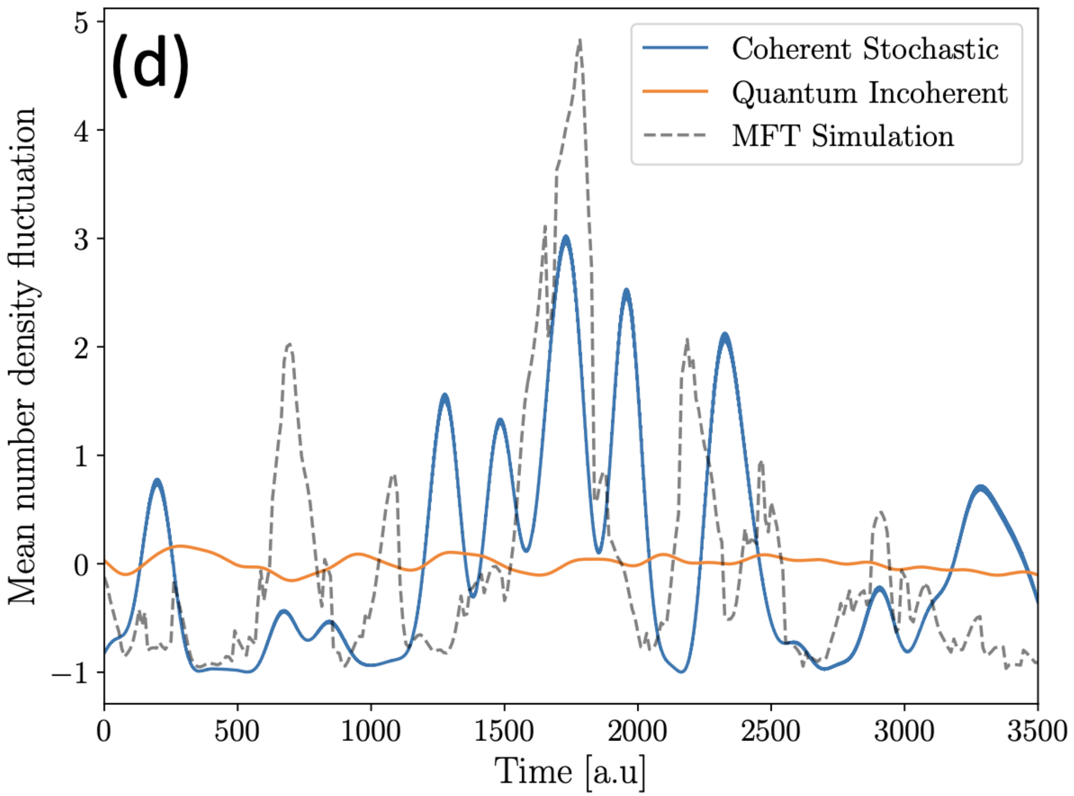
Open question: what is the standard deviation of thermal coherent state under realistic bandwidth of our quantum detector?
→ This could be Poisson in the end…

# Outline

- Background: classical to quantum detection
- Motivation of this talk: classical to quantum??
- Rigorous proof of classical to quantum
- Applications: coherence & homodyne with polarization
- Conclusion

# Temporal coherence of axion DM




D. Marsh arXiv:2211.13602

<u>Coherent stochastic model</u>

$$\phi(t) = \frac{\sqrt{2\rho_{\rm loc}}}{m}\sum_{i=0}^{i_{\rm max}}\alpha_i \cos\left[\left(1+\frac{v_i^2}{2}\right)mt+\delta_i\right]$$

<u>Quantum incoherent model</u>

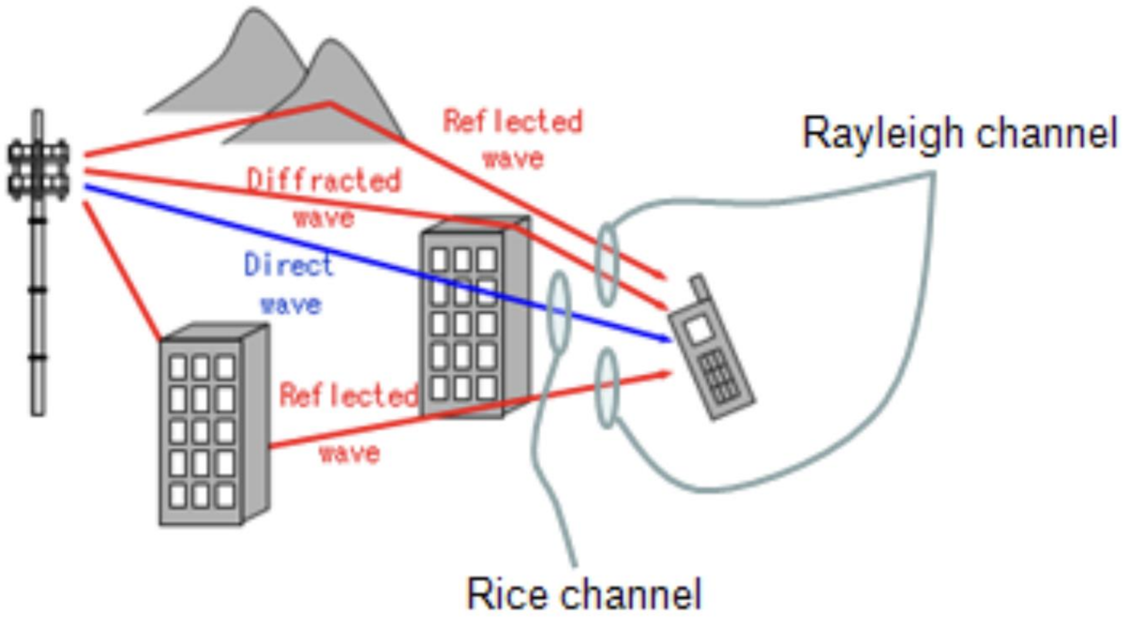
$$n = \frac{1}{2}m\langle|\psi_k|^2\rangle_k$$

$$\psi_k = \frac{\sqrt{2\rho_{\rm loc}}}{m}\sum_{i=0}^{i_{\rm max}}\alpha_i \exp\left[i\left(m\frac{v_i^2}{2}t+\delta_i+\varepsilon_{ik}\right)\right]$$

DM axion is in quantum coherent state ~ truly classical wave

# Coherent stochastic: Rayleigh fading

Random phase in classical microwaves → temporal coherency



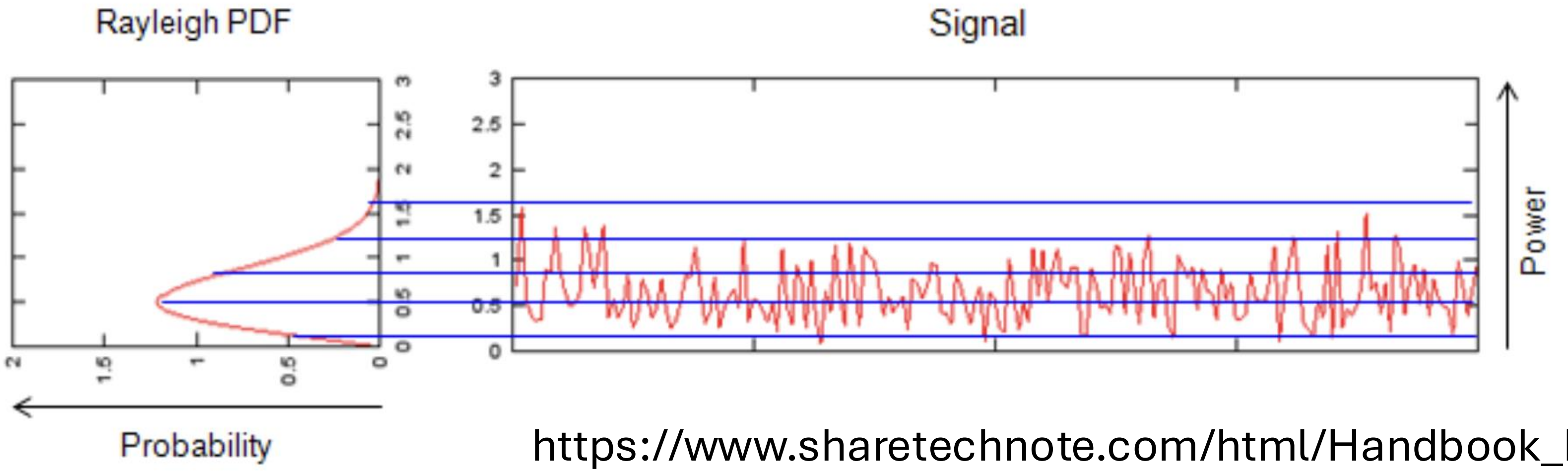

Source signal

$$v_s(t) = \cos(\omega t + \phi)$$

***Tele-communication***

Received signal

$$v_r(t) = \sum_{n=1}^{N} c_n \cos(\omega t + \phi + \phi_n)$$




Rayleigh distribution

$$P(x) = 2xe^{-x^2}\,\mathrm{d}x$$

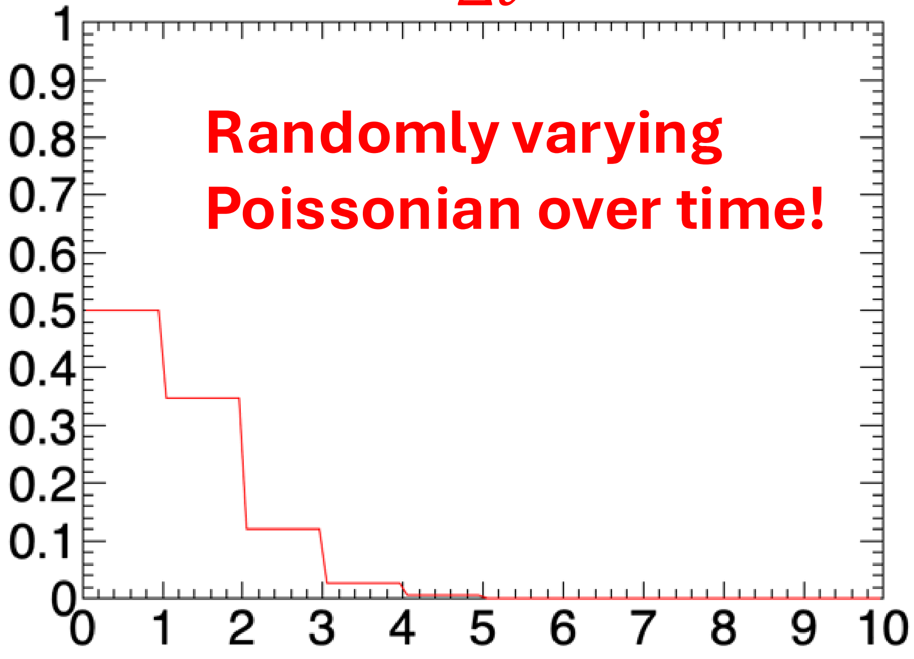
https://www.sharetechnote.com/html/Handbook_LTE_ChannelModel_Rayleigh.html

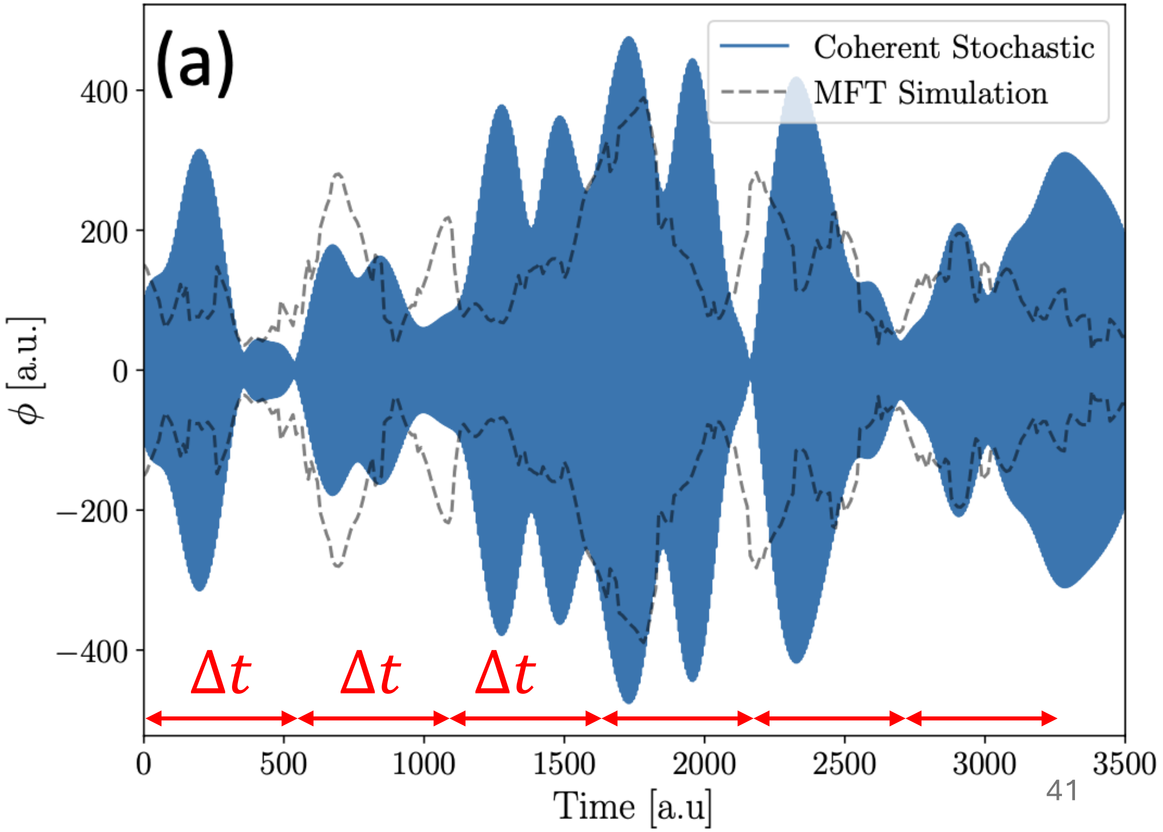
# What happens to the quantum detector response

$$P_n = |\langle n|\psi(t)\rangle|^2 = \frac{e^{-|\beta(t)|^2}|\beta(t)|^{2n}}{n!}$$

$$\beta(t) = \exp(-i\omega t)\underbrace{\boxed{\int_0^t}}_{\Delta t} dt'\phi(t')\exp(i\omega t')$$

- The photon number obeys Poisson distribution of varying mean
- Depends on integral of classical axion amplitude over temporal coherency
- $\Delta t$ → response time of quantum sensor? Or DAQ?



# Modulation in ***classical*** axion amplitude

$$\hat{H} = \hbar\omega\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) + \hbar\left[f(t)\hat{b}^\dagger + f^*(t)\hat{b}\right] \qquad f(t) = g_{a\gamma\gamma}a(t)\int(\boldsymbol{E}(x)\cdot\boldsymbol{B}_0)d^3x$$

Not only intrinsic axion amplitude's temporal variation, one can manipulate modulation

<u>Idea 1 (ADMX is also thinking)</u>

$$g_{a\gamma\gamma}a(t)\int(\boldsymbol{E}(x)\cdot\boldsymbol{B}_0\cos(\omega_{mod}t))d^3x$$

- Modulation in magnetic field
- Speed would be limited in SC magnet

<u>Idea 2 (MADMAX original)</u>

$$\hbar\left[f(t)\hat{b}^\dagger + f^*(t)\hat{b}\right]P\cos(\omega_{mod}t)$$

- Modulate polarization of the photons

→ Basic theory for homodyne detection (Lock-in method)

→ Enhance S/N even though losing bandwidth in quantum detection

# Modulation in ***classical*** axion amplitude

$$\hat{H} = \hbar\omega\left(\hat{b}^{\dagger}\right. \quad \cdots \quad \left.\right) \cdot \boldsymbol{B}_0)d^3x$$

Not only intrins … te modulation

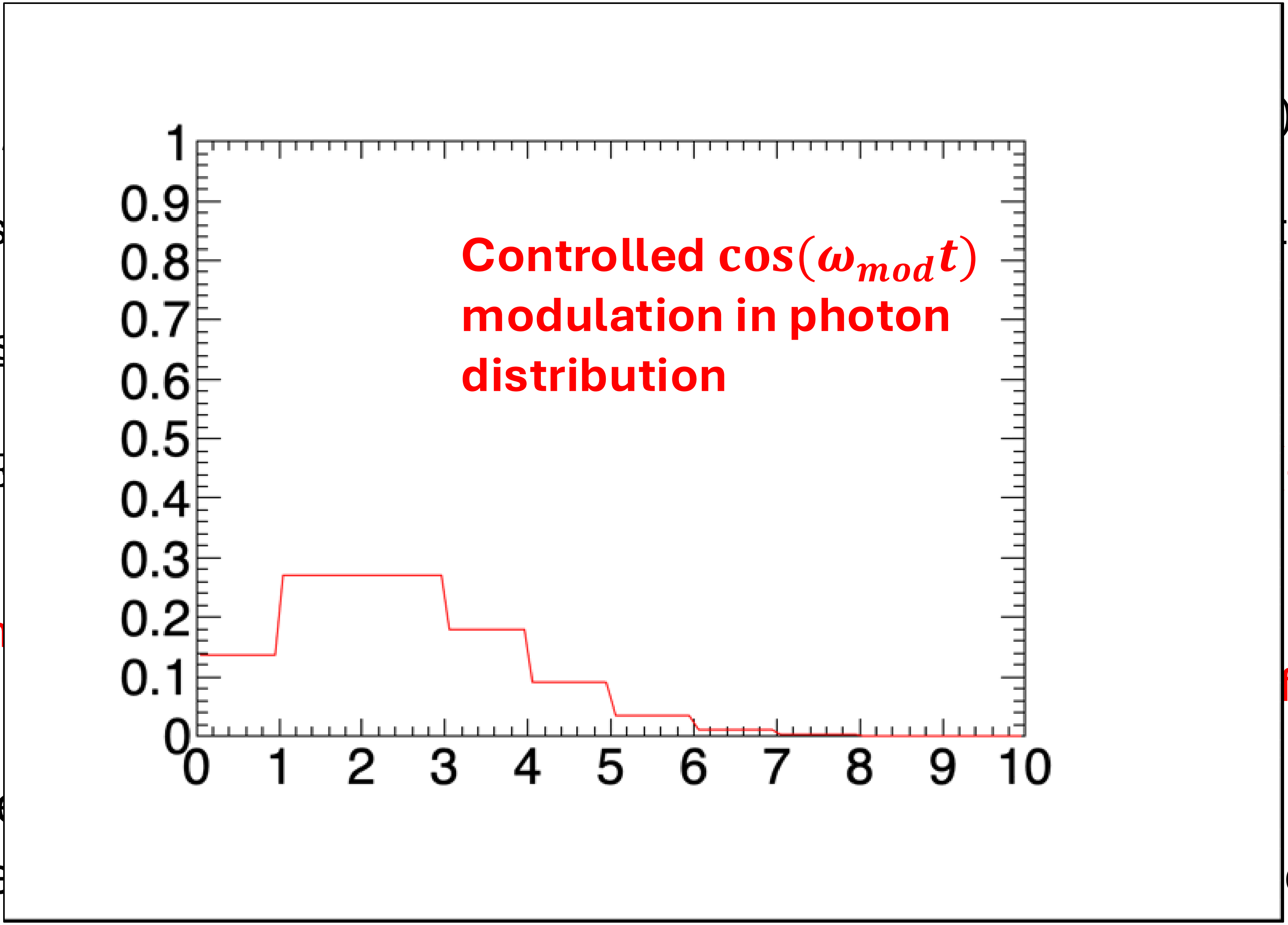
<u>Idea 1 (ADMX is</u>

$$g_{a\gamma\gamma}a(t)\int(\boldsymbol{E} \quad \cdots \quad \omega_{mod}t)$$

- Modulation …
- Speed wou … f the photons

→ Basic the …

→ Enhance … ection



# Outline

- Background: classical to quantum detection
- Motivation of this talk: classical to quantum??
- Rigorous proof of classical to quantum
- Applications: coherence & homodyne with polarization
- Conclusion

# Conclusion and outlook

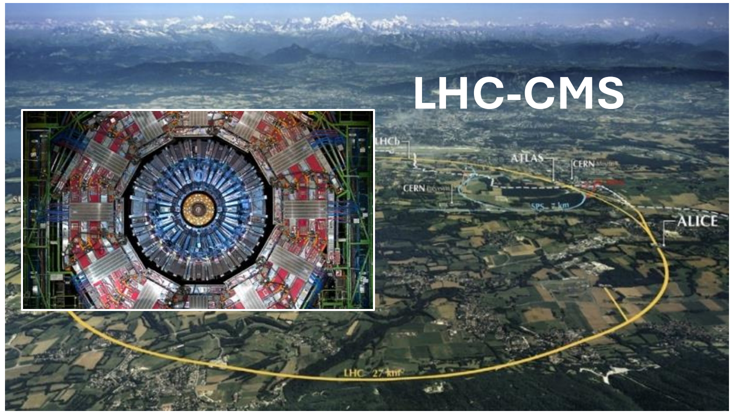
- Any classical axion field generates quantum coherent states of microwave photons
  - Rigorous proof with little assumptions (classicality of DM axions)
  - In photon counting type experiment, Poisson distribution of varying mean number of photons appear
  - The quantum state of our system is “coherent thermal state” ≠ coherent signal + thermal noise (standard deviation of photon numbers is not a naïve sum)
- The above theorem ensures classical argument in time variation of axion signal
  - Temporal coherency of axion amplitude
  - Artificial modulation → homodyne detection
- To be answered
  - Projective measurement → realistic quantum model of detectors
  - Detector requirement → material property, what to measure?
  - Is the assumption in classical axions truly reasonable?
  - How to optimize the time integral → depending on coherence time of axion?
  - Can we reconstruct time-varying photon distribution via finite sampling of a photon detector?
- To be developed
  - Correct Monte Carlo simulation
  - Response time, detection efficiency of the detectors

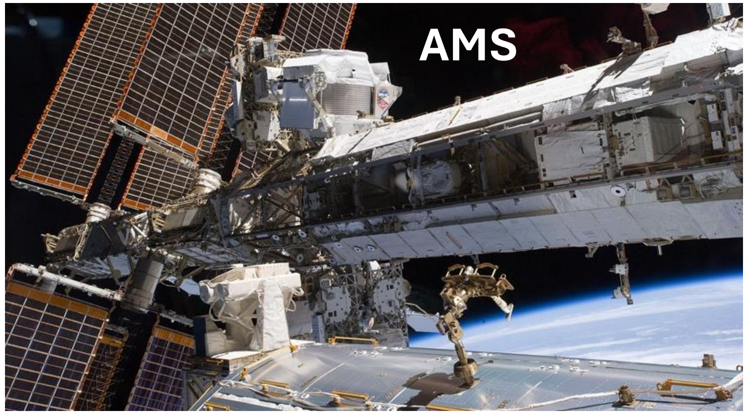
# backup

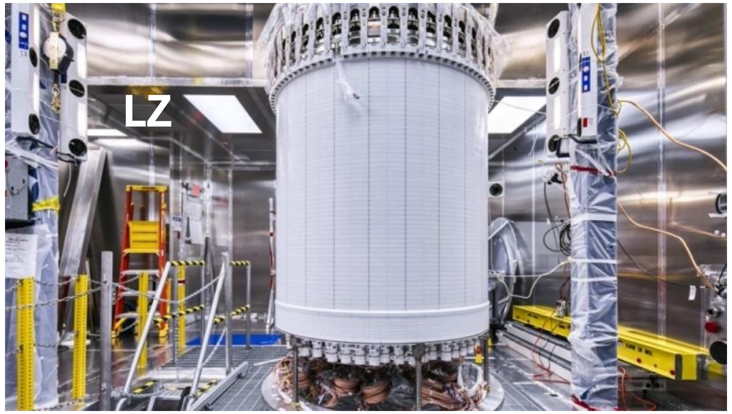
# Three ways to study dark matter candidates

Production in lab | Signal from astrophysics | DM from galaxy halo

WIMP
SUSY

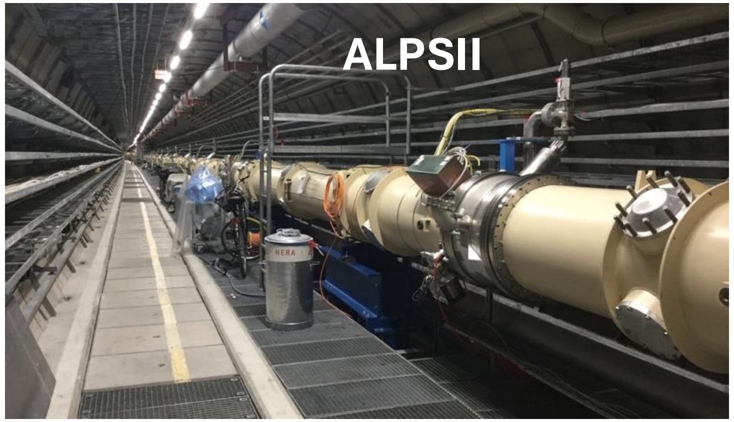



→ Common techniques: **particle** detection, reconstruction, PID, etc

WISP
axions






full knowledge in source | uncertainty in astrophysical models | Uncertainty in cosmological models

→ Common techniques: magnets & **photon science**

# Quantum electrodynamics (single mode)

With creation annihilation operators

$$\begin{cases} \hat{H} = \left(\hat{n} \boxed{+ \frac{1}{2}}\right)\hbar\omega = \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)\hbar\omega \\ \hat{\mathcal{E}}(\boldsymbol{r}, t) = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left[\hat{a}e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} - \hat{a}^\dagger e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}\right] \end{cases}$$

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1$$

$$[\hat{a}, \hat{a}] = \left[\hat{a}^\dagger, \hat{a}^\dagger\right] = 0$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

With Hermitian quantized amplitudes

$$\begin{cases} \hat{H} = \left(\hat{n} + \frac{1}{2}\right)\hbar\omega = [(\hat{x}_1)^2 + (\hat{x}_2)^2]\hbar\omega \\ \hat{\mathcal{E}}(\boldsymbol{r}, t) = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}[\hat{x}_1 \sin(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}) - \hat{x}_2 \cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})] \end{cases}$$

$$[\hat{x}_1, \hat{x}_2] = \frac{i}{2}$$

$$[\hat{x}_1, \hat{x}_1] = [\hat{x}_2, \hat{x}_2] = 0$$

$$\xleftrightarrow[\hat{a}^\dagger \equiv \hat{x}_1 - i\hat{x}_2]{\hat{a} \equiv \hat{x}_1 + i\hat{x}_2}$$

Uncertainty relations

General displacement operators → The Cauchy-Schwartz's inequality leads to Schrodinger's uncertainty relation

$$\Delta\hat{A} = \hat{A} - \langle\phi|\hat{A}|\phi\rangle$$

$$\Delta\hat{B} = \hat{B} - \langle\phi|\hat{B}|\phi\rangle$$

$$\underbrace{\langle\phi|\Delta\hat{A}^2|\phi\rangle}_{\text{dispersion}}\underbrace{\langle\phi|\Delta\hat{B}^2|\phi\rangle}_{\text{dispersion}} \geq \frac{1}{4}\left|\langle\phi|\{\Delta\hat{A}, \Delta\hat{B}\}|\phi\rangle\right|^2 + \frac{1}{4}\left|\langle\phi|[\hat{A}, \hat{B}]|\phi\rangle\right|^2$$

Applying to the amplitude operators leads to

Positive real number

$$\sigma_1^2\sigma_2^2 \equiv \langle\phi|\Delta\hat{x}_1{}^2|\phi\rangle\langle\phi|\Delta\hat{x}_2{}^2|\phi\rangle \geq \frac{1}{4}|\langle\phi|\{\Delta\hat{x}_1, \Delta\hat{x}_2\}|\phi\rangle|^2 + \frac{1}{4}|\langle\phi|[\Delta\hat{x}_1, \Delta\hat{x}_2]|\phi\rangle|^2 = \frac{1}{4}|\langle\phi|\{\Delta\hat{x}_1, \Delta\hat{x}_2\}|\phi\rangle|^2 + \frac{1}{16}$$

$\rightarrow \sigma_1\sigma_2 \geq \frac{1}{4}$: standard deviations of amplitude operators obeys this uncertainty relation

# Minimum uncertainty states $|\phi_{min}\rangle$

The minimum uncertainty

$$\sigma_1 \sigma_2 = \frac{1}{4}$$

can be obtained if

$$\begin{cases} \Delta\hat{x}_2|\phi_{min}\rangle = \mathrm{z}\Delta\hat{x}_1|\phi_{min}\rangle \ (z \in \mathbb{C}) \\ \langle\phi_{min}|\{\Delta\hat{x}_1, \Delta\hat{x}_2\}|\phi_{min}\rangle = 0 \end{cases}$$ From Cauchy-Schwartz

are applied. These conditions give

$$0 = \langle\phi_{min}|\Delta\hat{x}_1\Delta\hat{x}_2|\phi_{min}\rangle + \langle\phi_{min}|\Delta\hat{x}_2\Delta\hat{x}_1|\phi_{min}\rangle = (z + z^*)\langle\phi_{min}|\Delta\hat{x}_1^2|\phi_{min}\rangle$$
$$\rightarrow z + z^* = 0 \rightarrow z = i\lambda \ (\lambda \in R)$$

Substitute this condition to $z$

$$\Delta\hat{x}_2|\phi_{min}\rangle = i\lambda\Delta\hat{x}_1|\phi_{min}\rangle$$

From the definition

$$(\hat{x}_2 - \langle\phi_{min}|\hat{x}_2|\phi_{min}\rangle)|\phi_{min}\rangle = \mathrm{i}\lambda(\hat{x}_1 - \langle\phi_{min}|\hat{x}_1|\phi_{min}\rangle)|\phi_{min}\rangle$$
$$\rightarrow (\lambda\hat{x}_1 + i\hat{x}_2)|\phi_{min}\rangle = [\lambda\langle\phi_{min}|\hat{x}_1|\phi_{min}\rangle + i\langle\phi_{min}|\hat{x}_2|\phi_{min}\rangle]|\phi_{min}\rangle$$

If we select $\lambda = 1$, we can obtain one of the minimum uncertainty states $|\alpha\rangle$:

$$(\hat{x}_1 + i\hat{x}_2)|\phi_{min}\rangle = [\langle\phi_{min}|\hat{x}_1|\phi_{min}\rangle + i\langle\phi_{min}|\hat{x}_2|\phi_{min}\rangle]|\phi_{min}\rangle$$

$$\rightarrow \boxed{\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \ (\alpha \in \mathbb{C})}$$

as an eigenstate of the annihilation operator $\hat{a}$, which minimizes the uncertainty of the amplitude operators $\hat{x}_1$ and $\hat{x}_2$

# Coherent state $|\alpha\rangle$

$|\alpha\rangle$ is called coherent state. Its explicit representation with number states can be given by

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

and applying $\hat{a}$ gives

$$\alpha \sum_{n=0}^{\infty} c_n |n\rangle = \alpha|\alpha\rangle = \hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} c_n \hat{a}|n\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n}|n-1\rangle = \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}|n\rangle$$

Comparing the coefficients gives

$$c_{n+1} = \frac{\alpha}{\sqrt{n+1}} c_n$$

The coefficient can be recursively determined

$$c_n = \frac{\alpha}{\sqrt{n}} c_{n-1} = \frac{\alpha}{\sqrt{n}} \frac{\alpha}{\sqrt{n-1}} c_{n-2} = \frac{\alpha}{\sqrt{n}} \frac{\alpha}{\sqrt{n-1}} \cdots \frac{\alpha}{\sqrt{1}} c_0 = \frac{\alpha^n}{\sqrt{n!}} c_0$$

The normalization condition gives

$$1 = \langle\alpha|\alpha\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_m^* c_n \langle m|n\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\alpha^{*m}}{\sqrt{m!}} \frac{\alpha^n}{\sqrt{n!}} c_0^* c_0 \delta_{mn} = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2} \rightarrow |c_0| = e^{-|\alpha|^2/2}$$

Selecting one phase $c_0 = e^{-|\alpha|^2/2}$ leads to

$$\boxed{|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

Expectation value of # of photons

$$\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = \alpha^*\alpha\langle\alpha|\alpha\rangle = |\alpha|^2 \equiv \mu$$

# Projection of $|\alpha\rangle$ onto $|x_1\rangle$ returns a Gaussian

$$\langle x_1|\alpha\rangle = \frac{1}{\alpha}\langle x_1|\alpha|\alpha\rangle = \frac{1}{\alpha}\langle x_1|\hat{a}|\alpha\rangle = \frac{1}{\alpha}\langle x_1|(\hat{x}_1 + i\hat{x}_2)|\alpha\rangle = \frac{1}{\alpha}\int dx_1'\,\langle x_1|(\hat{x}_1 + i\hat{x}_2)|x_1'\rangle\langle x_1'|\alpha\rangle$$

<u>"Position" representation (Schrodinger representation)</u>

Wave function: $\Psi(x_1) \equiv \langle x_1|\alpha\rangle$

"Position" operator: $\langle x_1|\hat{x}_1|x_1\rangle = x_1$

"Momentum" operator: $[\hat{x}_1, \hat{x}_2] = \frac{i}{2} \to \langle x_1|\hat{x}_2|x_1\rangle \equiv \frac{1}{2i}\frac{d}{dx_1}$

<u>Schrodinger equation (in a narrow sense…)</u>

$$\longrightarrow \quad \alpha\Psi(x_1) = \left[x_1 + \frac{1}{2}\frac{d}{dx_1}\right]\Psi(x_1)$$

***Eigenvalue problem***

<u>A particular solution of this 1st order linear ordinary differential equation is Gaussian</u>

$$\Psi(x_1) = \exp(Ax_1^2 + Bx_1 + C)$$

$$\frac{d\Psi(x_1)}{dx_1} = [2Ax_1 + B]\Psi(x_1)$$

$$\longrightarrow \quad A = -1, \quad B = 2\alpha \quad \longrightarrow \quad \Psi(x_1) = \frac{1}{\sqrt{2\pi}}e^{-(x_1-\alpha)^2}$$

<u>The same would be applied for the "momentum" representation</u>

Wave function: $\Phi(x_2) \equiv \langle x_2|\alpha\rangle$

"Position" operator: $\langle x_2|\hat{x}_2|x_2\rangle = x_2$

"Momentum" operator: $[\hat{x}_1, \hat{x}_2] = \frac{i}{2} \to \langle x_2|\hat{x}_1|x_2\rangle \equiv \frac{i}{2}\frac{d}{dx_2}$

$$\alpha\Phi(x_2) = \left[\frac{i}{2}\frac{d}{dx_2} + ix_2\right]\Phi(x_2)$$

$$\longrightarrow \quad \Phi(x_2) = \frac{1}{\sqrt{2\pi}}e^{-(x_2+i\alpha)^2}$$

# Projection of $|\alpha\rangle$ onto $|n\rangle$ returns a Poisson distribution

When one performs photon counting experiment (for example with PMT), probability of observing n photon is

$$P(n) = |\langle n|\alpha\rangle|^2 = \left| e^{-|\alpha|^2/2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} \underbrace{\langle n|m\rangle}_{\delta_{nm}} \right|^2 = \left| e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right|^2 = \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2}$$

This is a Poisson distribution with mean number of photons $\mu = |\alpha|^2$

$$P(n;\mu) = \frac{\mu^n}{n!} e^{-\mu}$$

E. H. Bellamy, NIMA 339 468-476 (1994)

It is well-known that we observed a Poisson distribution from LED, convoluted with a response function of dynodes in PMT

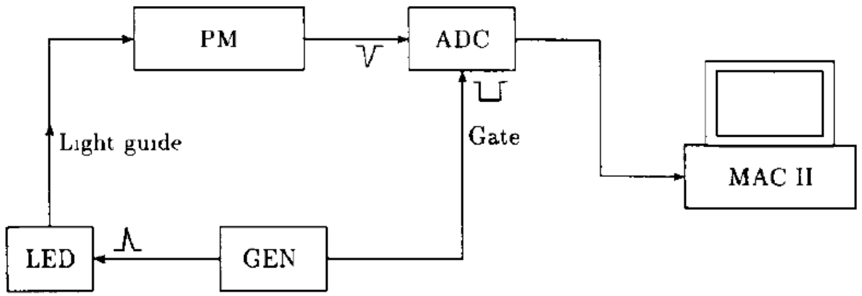



Fig. 1. Block scheme of the calibration setup.

$$S_{\text{ideal}}(x) = P(n;\mu) \otimes G_n(x)$$

$$= \sum_{n=0}^{\infty} \frac{\mu^n e^{-\mu}}{n!} \frac{1}{\sigma_1\sqrt{2\pi n}} \exp\left(-\frac{(x-nQ_1)^2}{2n\sigma_1^2}\right)$$

$$S_{\text{real}}(x) = \int S_{\text{ideal}}(x') B(x-x')\, dx'$$