Order α^2 Corrections to the Decay Rate of Orthopositronium

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Order α^2 corrections to the decay rate of orthopositronium are calculated in the framework of nonrelativistic QED. The resulting contribution is found to be in significant disagreement with one set of experimental measurements, though another experiment is in agreement with theory.

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The discrepancy between theory and experiment for the decay rate of orthopositronium has long been one of the outstanding problems in precision QED. The theory is given by

$$\Gamma_{o\text{-Ps}} = \Gamma_0\Big[1 + A\frac{\alpha}{\pi} + \frac{\alpha^2}{3}\ln\alpha + B\Big(\frac{\alpha}{\pi}\Big)^2 - \frac{3\alpha^3}{2\pi}\ln^2\alpha\Big], \quad (1)$$

where the lowest order decay rate is given by

$$\Gamma_0 = \frac{2}{9}(\pi^2 - 9)\frac{m\alpha^6}{\pi}. \quad (2)$$

The value of Γ_0 is 7.211 169 μs^{-1}. This is about 2.3% above the experimental determinations at Ann Arbor of

$$\Gamma_{o\text{-Ps}}(\text{gas}) = 7.0514(14)\ \mu s^{-1} \quad (3)$$

(Ref. [1]),

$$\Gamma_{o\text{-Ps}}(\text{vacuum}) = 7.0482(16)\ \mu s^{-1} \quad (4)$$

(Ref. [2]), and 2.4% above the somewhat less precise Tokyo measurement

$$\Gamma_{o\text{-Ps}}(SiO_2) = 7.0398(29)\ \mu s^{-1} \quad (5)$$

(Ref. [3]). The great bulk of this difference is accounted for by the one-loop correction [4], which has been evaluated with high accuracy in [5] to be

$$A = -10.286\,606(10). \quad (6)$$

Including this -2.39% effect along with the logarithmic terms of order $\alpha^2\Gamma_0$ [6] and $\alpha^3\Gamma_0$ [7], which contribute -0.01% and -0.0004%, respectively, gives a decay rate of 7.038 202 μs^{-1}. The remaining difference with the Ann Arbor experiments, with the 2.3% difference reduced to -0.1%, requires a rather large positive value for B [339(36) for the gas experiment and 257(41) for the vacuum experiment], which is the discrepancy mentioned above. The Tokyo experiment, on the other hand, which disagrees with the first two by several standard deviations, is consistent with a small value. While the experimental situation clearly requires more work, it is also obviously important to directly evaluate the constant B.

While the need for calculating B has been clear for two decades, there are two difficulties that have prevented its evaluation until now. The first is simply the large number (83) of diagrams that contribute, many of which have two-loop ultraviolet infinities. More importantly, a number of these diagrams have a serious kind of infrared divergence associated with the fact that positronium is a bound state.

These problems have recently been overcome for the case of parapositronium decay [8]. The main theoretical tool used in this work is nonrelativistic quantum electrodynamics (NRQED) [9]. This approach allows the high-energy part of the problem to be treated as an on-shell scattering process. In this case the complications of the bound state are not present, and Feynman gauge can be used. The low-energy, bound state part can be treated in Coulomb gauge with a small set of operators that describe relativistic and QED corrections to a Schrödinger problem. A matching procedure carried out with free particle scattering amplitudes then allows the two parts to be combined into a complete calculation.

The present calculation, while similar in spirit to that of the parapositronium calculation, regulates long wavelength singularities by giving the photon a small mass $m_e\lambda$: Ref. [8] instead uses dimensional regularization. We have chosen to use a slight variation of a recent NRQED calculation of one-photon annihilation contributions to ground state positronium hyperfine splitting [10] that uses a photon mass, as it is easily generalized to the decay rate calculation.

In NRQED the annihilation of orthopositronium can be accounted for by modifying the amplitude for one-photon annihilation, $2\pi\alpha/m^2$, to

$$V_4(\vec{k},\vec{l}) = \frac{2\pi\alpha}{m^2}\Big(1 - \frac{4i\alpha^2(\pi^2 - 9)}{9\pi}\Big). \quad (7)$$

The independence of V_4 on the momentum is a reflection

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of the fact that annihilation occurs, on an atomic scale, nearly at a point in coordinate space. At the level of precision required here, we will also need to consider a modification that accounts for the interaction not being exactly pointlike [11],

$$V_4^{\rm der}(\vec{k},\vec{l}) = \frac{i\alpha^3(\pi^2-9)X}{27m^4}(\vec{k}^2+\vec{l}^2), \qquad (8)$$

where $X = (19\pi^2 - 132)/(\pi^2 - 9)$.

In first-order perturbation theory, taking the expectation value of V_4 (which corresponds to multiplying by the square of the wave function at the origin $m^3\alpha^3/8\pi$) and using $\Gamma = -2\,\mathrm{Im}(E)$ reproduces Eq. (2). In addition to this amplitude other operators accounting for relativistic effects are present [10], and lead to the following ultraviolet divergent expression in second-order Rayleigh-Schrödinger perturbation theory, which has the first-order effect of $V_4^{\rm der}$ included:

$$\Gamma_{\rm NRQED} = \Gamma_0\left[1 + \frac{\alpha}{\pi}e_1 + \left(\frac{\alpha}{\pi}\right)^2 e_2\right] + \alpha^2\Gamma_0\left[-\frac{8\Lambda}{3\pi\alpha} - \frac{1}{3}\ln\frac{\Lambda}{\alpha} - \frac{11}{24} - \frac{X}{12\pi}\left(\frac{2\Lambda}{\alpha} - \frac{3\pi}{4}\right)\right]. \qquad (9)$$

Here we have renormalized the imaginary part of V_4 with a power series in α and introduced an ultraviolet cutoff $m\Lambda$ on the momentum space integrations.

The constants e_1 and e_2 are determined by requiring that the amplitude for free particle scattering at threshold in NRQED be equal to that determined in a complete QED calculation. The one-loop QED calculation at threshold has an amplitude corresponding to the decay rate

$$\Gamma_1 = \frac{\alpha}{\pi}\Gamma_0\left[\frac{2\pi}{\lambda} + A(\lambda)\right], \qquad (10)$$

where $A(\lambda) = -10.286\,60 + 15.39\lambda$. Even though the limit $\lambda \to 0$ is taken at the end of the calculation, we keep terms of order λ in the one-loop calculation because some terms enter the two-loop calculations with a factor $1/\lambda$. The one-loop matching calculation then allows us to determine

$$e_1 = \frac{8\Lambda}{3} + \frac{X\Lambda}{6} + A(\lambda) - \frac{7\pi\lambda}{12} - \frac{X\pi\lambda}{12}. \qquad (11)$$

If we further define the two-loop QED decay rate as

$$\Gamma_2 = \left(\frac{\alpha}{\pi}\right)^2\Gamma_0\left[\frac{(2\ln 2 + 1)\pi^2}{\lambda^2} + \frac{2\pi A(\lambda)}{\lambda} + \frac{\pi^2}{3}\ln\lambda + B_2\right], \qquad (12)$$

the two-loop matching calculation gives

$$e_2 = \frac{\pi^2}{3}\ln\Lambda - \frac{\pi^2 X}{24} + \frac{11\pi^2}{6} - \frac{2\pi^2\ln 2}{3} + B_2. \qquad (13)$$

The reason for defining Γ_2 in terms of $A(\lambda)$ rather than the physical limit $A(0)$ is a practical one, associated with subtraction schemes required to deal with the most infrared divergent two-loop diagrams. It also has the advantage of leading to an exact cancellation with the factor $A(\lambda)$ in e_1 in the matching calculation.

With this determination of e_1 and e_2 the NRQED decay rate becomes ultraviolet and infrared finite, and is given by

$$\Gamma_{\rm NRQED} = \Gamma_0\left\{1 + \frac{\alpha}{\pi}A(0) + \left(\frac{\alpha}{\pi}\right)^2 \times\left[\frac{\pi^2}{3}\ln\alpha + \pi^2\left(\frac{11}{8} - \frac{2\ln 2}{3}\right) + \pi^2\frac{X}{48} + B_2\right]\right\}, \qquad (14)$$

and numerically the constant we wish to determine is given by $B = B_2 + \pi^2(0.9129 + 1.3302) = B_2 + 22.14$. We note that the constant 0.9129 differs from Ref. [11], where it is given as 1.16, the numerical value of $\frac{13}{8} - \frac{2}{3}\ln(2)$. An additional contribution of $-\frac{1}{4}$ to this number has recently been found [12], removing the discrepancy. In addition, Hill and Lepage [13] have recently recalculated a number of QED effects in a new nonperturbative implementation of NRQED, and obtain 0.9125(5), so all NRQED calculations are now in agreement. As an additional check of the method, we verified that our implementation of NRQED, when applied to the one-photon annihilation contribution to ground state hyperfine splitting, reproduces the known answer [10].

To finish the calculation, the free two-loop QED calculation must be carried out and B_2 extracted. While the two-loop calculation is sufficiently involved that we must defer its detailed description to a longer work [14], it is useful to refer to the diagrams that enter the one-loop calculation. We refer to the six diagrams of Fig. 1 as the outer vertex (OV), inner vertex (IV), double vertex (DV), self-energy (SE), ladder (L), and annihilation (A)

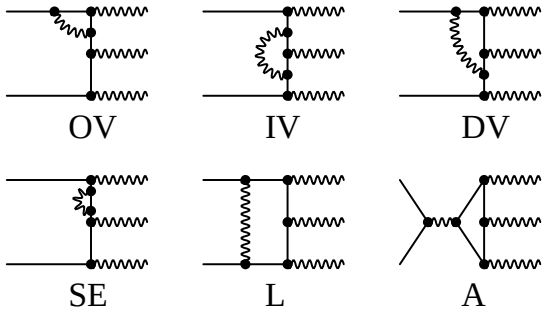

FIG. 1. Diagrams contributing to the one-loop decay rate correction.

TABLE I. Renormalized one-loop contributions to the orthopositronium decay rate.

Contribution	$\frac{\pi}{\lambda}(\frac{\alpha}{\pi})\Gamma_0$	$\ln\lambda(\frac{\alpha}{\pi})\Gamma_0$	$\frac{\alpha}{\pi}\,\Gamma_0$	$\lambda\,\frac{\alpha}{\pi}\,\Gamma_0$
$\Gamma_{\rm OV}$	0	−4	−1.028 861 425	1.8756(1)
$\Gamma_{\rm IV}$	0	−2	−1.839 322 925	4.7124
$\Gamma_{\rm DV}$	0	0	−3.567 629(21)	7.5499(2)
$\Gamma_{\rm SE}$	0	4	4.784 983 909	−11.0445(1)
$\Gamma_{\rm L}$	2	2	−7.821 768(32)	12.296(4)
$\Gamma_{\rm A}$	0	0	−0.814 057 3(3)	0.0
Total	2	0	−10.286 606(10)	15.389(4)

contributions. After the ultraviolet divergences are removed by renormalization, the individual values of the diagrams are presented in Table I.

The 83 QED diagrams that contribute to the decay rate at two-loop order break into 11 classes that we label (a)–(k). Class (a) consists of irreducible two-loop vertex corrections which generalize the OV diagrams. While free of binding singularities, their evaluation is complicated by the need to carry out two-loop renormalization. Similar comments apply to class (b), the two-loop generalization of the IV diagram, and class (c), the generalization of the SE diagram. We regulate ultraviolet divergences by using dimensional regularization, working in $n = 4 - 2\epsilon$ dimensions, and the finite photon mass regulates infrared divergences. Renormalization constants in this scheme have not, to our knowledge, been presented in the literature: details of their calculation will be given elsewhere [15].

Class (d) consist of diagrams with reducible two-loop corrections, in which two separated ultraviolet divergent one-loop corrections are present. The next set of diagrams, which have no ultraviolet or infrared singularities, are those of class (e), which generalize the DV diagrams. The most difficult to evaluate diagrams were in the (f) class, which generalize the L diagram. Most of these contributions diverge as $1/\lambda$, and the most singular as $1/\lambda^2$. Canceling $\ln\lambda/\lambda$ divergences characteristic of Feynman gauge were present that were quite difficult to handle numerically.

Class (g) consists of nine diagrams in which the DV diagram has ultraviolet divergent one-loop radiative corrections in all possible vertices and propagators. Class (h) consists of radiative corrections to the A diagram, and have previously been calculated in Ref. [16]. Because that calculation used a Bethe-Salpeter formalism, a $\ln\alpha$ was present that has to be replaced with a $\ln\lambda$ in our present formalism: the additive constant, however, is unchanged. Class (i) consists of diagrams where a vacuum polarization loop has been inserted in all possible places in the one-loop calculation. These have also been previously treated [17] and [18], as has the last two-loop effect we include, the square of the one-loop amplitude, which we call class (j) [5,19]. As with class (h), our present formalism leads to terms that depend on λ, but the additive constant is again unchanged.

Finally, class (k), which involves two of the three photons emitted in the decay undergoing light-by-light scattering, has not been calculated: because of the small numerical contributions of these diagrams in parapositronium [8] we consider it highly unlikely that the omission of these diagrams will affect our conclusions.

The results of the calculation are summarized in Table II.

We see from Table II that $B_2 = 22.38(26)$, so the complete result for B is 44.52(26). While this is indeed a relatively large contribution, it leaves the theoretical prediction,

TABLE II. Contributions to the orthopositronium decay rate by class.

Diagram	$\frac{\alpha^2}{\lambda^2}\,\Gamma_0$	$\frac{\alpha^2}{\pi\lambda}\,\Gamma_0$	$\ln\lambda\,\frac{\alpha}{\pi}$	$(\frac{\alpha}{\pi})^2\ln^2\lambda\Gamma_0$	$(\frac{\alpha}{\pi})^2\ln\lambda\Gamma_0$	$(\frac{\alpha}{\pi})^2\Gamma_0$
a	0	0	$-\Gamma_{\rm OV}$	−2	0	−5.618
b	0	0	$-\Gamma_{\rm IV}$	−1	0	−0.705
c	0	0	$-\Gamma_{\rm SE}$	2	0	0.058
d	0	0	0	0	0	2.421
e	0	0	0	0	0	9.259(9)
f	$2\ln2$	A	$\Gamma_{\rm SE+OV+IV+DV}$	1	$-2\pi^2/3$	−20.50(26)
g	0	0	$-\Gamma_{\rm DV}$	0	0	−1.372
h	0	0	0	0	π^2	9.007
i	1	A	0	0	0	0.965
j	0	0	0	0	0	28.860(2)
Total	$2\ln2 + 1$	2A	0	0	$\pi^2/3$	22.38(26)

$$\Gamma = 7.039\,934(10)\ \mu s^{-1}, \qquad (15)$$

well below the Ann Arbor results, by 8 and 5 standard deviations for the gas and vacuum measurements, respectively, though consistent with the Tokyo measurement. We note that the quoted theoretical error comes primarily from the evaluation of group (f), and does not include the uncertainty associated with the uncalculated group (k) terms.

No conclusions can be drawn until the experimental situation is clarified. If the Ann Arbor results are confirmed, while of course it is tempting to consider explaining this effect through exotic interactions [20], it is worth noting that there is also at present a significant discrepancy between theory and experiment in the ground state hyperfine splitting of positronium. While there were disagreements between various calculations for some time, recently complete agreement [13,21–23] on a value of 203 392.05 MHz has been found. This value is 4 experimental standard deviations above the Yale measurement [24] and 2.8 above the Brandeis measurement [25]. If the present disagreement of theory and experiment in positronium persists, any explanation in terms of new physics would be most compelling if both discrepancies were explained by the same mechanism.

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